

# The Dirac operator on compact symmetric spaces

Emiko Dupont

Master thesis for  
the Cand.Scient degree in mathematics  
at the University of Copenhagen  
Advisor: Henrik Schlichtkrull

**June 2003**



## Acknowledgements

Firstly, I wish to thank my advisor, Henrik Schlichtkrull, for his guidance throughout my work with this thesis and for introducing me to the field of representation theory in which I have benefitted greatly from his expertise. In addition, I am grateful for his advice and support during my application for a Ph.D. position.

I also wish to thank the mathematics department at the University of Utah where part of this work was carried out. In particular, I am grateful to Henryk Hecht for his generous efforts and guidance, and to Dragan Milićić for his advice. I have also benefitted from my stay at the University of Oxford during which I was introduced to the field of differential geometry. In particular, I would like to thank Ulrike Tillmann and Andrew Dancer.

Finally, I am grateful to my family and friends for their support and encouragement. Special thanks to my friends at the University of Copenhagen for creating a pleasant and friendly working environment throughout my studies, and to my parents, Johan Dupont and Mariko Hayashi, my sister, Yoko Dupont, and my fiancé, Alastair Craw, for putting up with me, especially in recent months.



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>The spinor representation</b>	<b>3</b>
2.1	Clifford algebras . . . . .	3
2.2	The spin group . . . . .	7
<b>3</b>	<b>Spin structure on <math>G/H</math></b>	<b>13</b>
3.1	The tangent bundle on $G/H$ . . . . .	13
3.2	The spinor bundle on $G/H$ . . . . .	14
<b>4</b>	<b>The Dirac operator</b>	<b>15</b>
4.1	Sections of induced bundles . . . . .	15
4.2	Connections on induced bundles . . . . .	17
4.3	The Dirac operator on $G/H$ . . . . .	19
<b>5</b>	<b>Symmetric spaces</b>	<b>22</b>
5.1	Basic notions . . . . .	22
5.2	The spinor representation . . . . .	26
5.3	The irreducible parts of $\chi$ . . . . .	29
<b>6</b>	<b>The Dirac operator on symmetric spaces</b>	<b>33</b>
6.1	The action of Casimir elements . . . . .	33
6.2	The square of the Dirac operator . . . . .	35
6.3	The representations $\tilde{\pi}_{\mu}^{\pm}$ . . . . .	41
<b>7</b>	<b>Examples</b>	<b>50</b>
7.1	$\mathrm{SO}(3)/\mathrm{SO}(2)$ . . . . .	50
7.2	$\mathrm{SO}(2m+1)/\mathrm{SO}(2m)$ . . . . .	52



## Abstract

Let  $G$  be a compact connected semisimple Lie group and let  $H \subset G$  be a closed connected subgroup such that  $\text{rank } G = \text{rank } H$  and  $G/H$  is a symmetric space. Given an irreducible representation of  $H$ , we define a Dirac operator  $D$  and determine the representations of  $G$  in the kernel of  $D$ . Moreover, we show that any irreducible representation of  $G$  can be constructed in this way. Our approach is similar to that of Parthasarathy [Par72].

## 1 Introduction

Let  $G$  be a compact connected semisimple Lie group and let  $H \subset G$  be a closed connected subgroup such that  $\text{rank } G = \text{rank } H$  and  $G/H$  is a symmetric space. For each irreducible representation  $(V, \tau)$  of  $H$  we define a Dirac operator  $D$  on the sections of a vector bundle  $\underline{S} \otimes V$  on  $G/H$ . The operator  $D$ , which depends on the representation  $(V, \tau)$ , splits into two parts  $D^+$  and  $D^-$ , and the kernels of  $D^\pm$  are finite-dimensional representations  $\tilde{\pi}^\pm$  of  $G$  under the left regular action. The objective of this paper is to show that one of the representations  $\tilde{\pi}^\pm$  is an irreducible representation of  $G$  while the other is zero (except in some cases when both are zero). Moreover, every irreducible representation of  $G$  can be constructed as either  $\tilde{\pi}^+$  or  $\tilde{\pi}^-$  for some irreducible representation  $(V, \tau)$  of  $H$ . This gives a geometric realization of the representations of  $G$  in terms of representations of  $H$ .

We now describe the contents of the paper in more detail. In sections 2-4 we review the theory necessary to define the Dirac operator and study its properties. Given an  $n$ -dimensional real vector space, one defines a representation  $(S, \sigma)$  of the spin group  $\text{Spin}(n)$  called the spinor representation which we investigate in section 2. Then in section 3 we look at (not necessarily compact or symmetric) homogeneous spaces  $G/H$ . In certain cases, the spinor representation induces a representation  $(S, \chi)$  of  $H$  which we also call the spinor representation, and we obtain an induced vector bundle  $\underline{S}$  on  $G/H$  called the spinor bundle. Thus if  $(V, \tau)$  is any representation on  $H$  we get a vector bundle  $\underline{S} \otimes V$  on  $G/H$ . Under additional assumptions on  $G$  and  $H$  we also have the half spinor representations  $(S^\pm, \chi^\pm)$  and induced bundles  $\underline{S}^\pm \otimes V$  on  $G/H$ . In section 4 we define the Dirac operator  $D$  and study some of its properties. By definition,  $D$  is an operator on the space of sections  $\Gamma(\underline{S} \otimes V)$  of  $\underline{S} \otimes V$ . Since  $D$  is an elliptic first order differential operator which commutes with the left regular action of  $G$  on  $\Gamma(\underline{S} \otimes V)$ , it follows that the kernel of the Dirac operator is a finite-dimensional representation  $\tilde{\pi}$  of  $G$  under the left regular action. If the half spinor representations exist, we also have the operators  $D^\pm$  on  $\Gamma(\underline{S}^\pm \otimes V)$  and we obtain finite-dimensional representations  $\tilde{\pi}^\pm$  of  $G$  on the kernel of  $D^\pm$ .

In sections 5-7 we specialize to the case of compact symmetric spaces  $G/H$  where  $\text{rank } G = \text{rank } H$ . By finding the irreducible parts of the spinor representation  $\chi$  of  $H$  in section 5, we determine the action of the Casimir element of  $H$  which

in turn is an important step in determining the square of the Dirac operator in section 4. The square of the Dirac operator consists of a constant term plus the action of the Casimir element of  $G$  under the left regular action. Thus elements of the kernel of  $D$  are eigenvectors of the Casimir operator and this gives a criterion which must be satisfied by the subrepresentations of  $\tilde{\pi}^\pm$ . As a consequence, the irreducible parts of  $\tilde{\pi}^\pm$  have multiplicity at most one and, furthermore,  $\tilde{\pi}^+$  and  $\tilde{\pi}^-$  have no common irreducible part. On the other hand, by calculating the difference between the trace of  $\tilde{\pi}^+$  and that of  $\tilde{\pi}^-$ , we see that  $\tilde{\pi}^+$  and  $\tilde{\pi}^-$  differ in at most one irreducible part. Hence one of  $\tilde{\pi}^+$  or  $\tilde{\pi}^-$  is irreducible and the other is zero (unless both are zero). The main result, theorem 6.6, then follows. Finally, in section 7, we study the specific case  $\mathrm{SO}(2m+1)/\mathrm{SO}(2m)$ .

For  $G/H$  a symmetric space where  $G$  is non-compact and  $H$  a maximally compact subgroup, Parthasarathy [Par72] uses representations on the kernels of Dirac operators  $D^\pm$  to construct the so-called discrete series of  $G$ , i.e., the representations of  $G$  which are equivalent to the left or right regular representation of  $G$  on a closed invariant subspace of  $L_2(G)$ . The proof of our main result, theorem 6.6, follows an approach similar to that of [Par72] but is simplified by the compactness assumption. More recent work by Slebarski [Sle87] considers the case where  $G$  is compact and  $H$  is a maximal torus of  $G$ , i.e.,  $G/H$  need not be symmetric. A result similar to theorem 6.6 holds but Slebarski's proof is complicated by the presence of an extra term in the square of the Dirac operator. This extra term depends on the torsion of the reductive connection; in the symmetric case the reductive connection is torsion-free.



## 2 The spinor representation

### 2.1 Clifford algebras

In this section we recall some properties of Clifford algebras. There is a classification of Clifford algebras which shows that they are in fact familiar matrix algebras or a direct sum of two matrix algebras. This simplifies their representation theory and it shows that Clifford algebras can be thought of as Lie algebras. Studying Clifford algebras will enable us to define the spinor representation of the spin group in section 2.2. The exposition is mainly based on Lawson and Michelsohn [LM89] and Gilbert and Murray [GM91].

Let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{F}$  (with  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ) with a quadratic form  $q$ . Let  $\text{Cl}(V, q)$  denote the (universal) Clifford algebra associated to  $(V, q)$ . This is an associative algebra with unit which is generated by the identity 1 and  $V$  subject to the relation

$$v \cdot v + q(v)1 = 0 \quad \text{for all } v \in V.$$

The Clifford algebra has the following universal property.

**Proposition 2.1.** *Let  $f: V \rightarrow A$  be a linear map into an associative algebra over  $\mathbb{F}$  with unit such that*

$$f(v) \cdot f(v) + q(v)1 = 0 \quad \text{for all } v \in V.$$

*Then  $f$  extends uniquely to an algebra homomorphism  $f: \text{Cl}(V, q) \rightarrow A$ .  $\text{Cl}(V, q)$  is the unique associative algebra over  $\mathbb{F}$  with this property.*

*Proof.* See proposition I.1.1 [LM89]. □

Let  $B$  be the bilinear form associated to  $q$ , i.e.,

$$B(v, w) = \frac{1}{2}(q(v) + q(w) - q(v - w)) \quad \text{for all } v, w \in V.$$

We say that  $(V, q)$  is non-degenerate if

$$V^\perp = \{w \in V \mid B(v, w) = 0 \text{ for all } v \in V\} = \{0\}.$$

In the following we assume that  $V$  is non-degenerate. Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $V$  in the sense that

$$B(e_i, e_j) = 0 \quad \text{for } i \neq j$$

$$q(e_i) \in \{\pm 1\}.$$

If  $\mathbb{F} = \mathbb{C}$  we may assume that  $q(e_i) = 1$ . The elements of the orthonormal basis satisfy the relations

$$e_j e_k + e_k e_j = -2q(e_j)\delta_{jk}$$

and the products

$$e_1^{m_1} \cdots e_n^{m_n}, \quad m_j \in \{0, 1\}$$

form a basis of  $\text{Cl}(V, q)$  (where  $e_1^0 \cdots e_n^0 = 1$ ). In particular,  $\text{Cl}(V, q)$  has dimension  $2^n$ .

If  $(V, q)$  is an  $n$ -dimensional quadratic space over  $\mathbb{R}$ , we can choose a basis of  $V \cong \mathbb{R}^n$  such that  $q$  is of the form

$$q_{r,s}(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2 \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

where  $r + s = n$ . We denote the Clifford algebra of  $(\mathbb{R}^{r+s}, q_{r,s})$  by  $\text{Cl}(r, s)$ . In the special case of  $s = 0$ , the associated bilinear form  $B$  is just the usual inner product on  $\mathbb{R}^n$  and we denote the Clifford algebra  $\text{Cl}(n, 0)$  by  $\text{Cl}(n)$ .

Similarly if  $(V, q)$  is an  $n$ -dimensional quadratic space over  $\mathbb{C}$ , we can choose a basis of  $V \cong \mathbb{C}^n$  such that  $q$  is of the form

$$q_{\mathbb{C}}(z) = z_1^2 + \cdots + z_n^2 \quad \text{for } z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

We denote the Clifford algebra of  $(\mathbb{C}^n, q_{\mathbb{C}})$  by  $\mathbb{C}\text{Cl}(n)$ . Now consider the complexification  $\text{Cl}(r, s) \otimes_{\mathbb{R}} \mathbb{C}$  of  $\text{Cl}(r, s)$ . By proposition 2.1 this is just the Clifford algebra of the complex quadratic space  $(\mathbb{C}^n, q_{r,s} \otimes 1)$  for  $r + s = n$  (where we think of  $\mathbb{C}^n$  as  $\mathbb{R}^{r+s} \otimes_{\mathbb{R}} \mathbb{C}$ ) and so

$$\mathbb{C}\text{Cl}(n) \cong \text{Cl}(r, s) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{for } r + s = n.$$

We claim that  $\text{Cl}(V, q)$  is a  $\mathbb{Z}_2$ -graded algebra. To see this, let  $\bar{\alpha}: V \rightarrow V$  be the linear map given by  $\bar{\alpha}(v) = -v$ . This extends by proposition 2.1 to a map  $\bar{\alpha}: \text{Cl}(V, q) \rightarrow \text{Cl}(V, q)$ . Since  $\bar{\alpha}^2 = \text{Id}$ , we get a splitting of  $\text{Cl}(V, q)$  into the  $+1$  and  $-1$  eigenspaces  $\text{Cl}(V, q)^0$  and  $\text{Cl}(V, q)^1$  which we call the even part and odd part of  $\text{Cl}(V, q)$  respectively. Note that  $\text{Cl}(V, q)^0$  is spanned by the even products  $\{e_1 \cdots e_{2k} \mid 2k \leq n\}$  and  $\text{Cl}(V, q)^1$  is spanned by the odd products  $\{e_1 \cdots e_{2k-1} \mid 2k-1 \leq n\}$ . This splitting is a  $\mathbb{Z}_2$ -grading of  $\text{Cl}(V, q)$  in the sense that

$$\text{Cl}(V, q)^0 \text{Cl}(V, q)^0 = \text{Cl}(V, q)^1 \text{Cl}(V, q)^1 = \text{Cl}(V, q)^0$$

$$\text{Cl}(V, q)^1 \text{Cl}(V, q)^0 = \text{Cl}(V, q)^0 \text{Cl}(V, q)^1 = \text{Cl}(V, q)^1.$$

Note that  $\text{Cl}(V, q)^0$  is a subalgebra of  $\text{Cl}(V, q)$ . In the case of  $\text{Cl}(n)$ , we get the following result:

**Proposition 2.2.** *There is an algebra isomorphism*

$$\text{Cl}(n) \cong \text{Cl}(n+1)^0.$$

*It follows that also*

$$\mathbb{C}\text{Cl}(n) \cong \mathbb{C}\text{Cl}(n+1)^0.$$

*Proof.* Let  $f: \mathbb{R}^n \rightarrow \text{Cl}(n+1)^0$  be given by

$$f(e_i) = e_i e_{n+1} \quad \text{for } i \in \{1, \dots, n\}.$$

This extends by proposition 2.1 to the desired algebra isomorphism  $\text{Cl}(n) \rightarrow \text{Cl}(n+1)^0$ . (See theorem I.3.7 of [LM89]).  $\square$

It turns out that the Clifford algebras  $\text{Cl}(r, s)$  and  $\mathbb{C}\text{l}(n)$  are familiar matrix algebras over  $\mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  (considered as algebras over  $\mathbb{R}$ ). We now focus our attention on the algebras  $\text{Cl}(n)$  and  $\mathbb{C}\text{l}(n)$  which are classified in the following theorem. (A similar classification of  $\text{Cl}(r, s)$  can be made).

**Theorem 2.3.** *For  $n \geq 0$  we have that*

$$\begin{aligned} \text{Cl}(n+8) &\cong \text{Cl}(n) \otimes_{\mathbb{R}} \text{Cl}(8) \\ \mathbb{C}\text{l}(n+2) &\cong \mathbb{C}\text{l}(n) \otimes_{\mathbb{C}} \mathbb{C}\text{l}(2). \end{aligned}$$

*Hence we get a classification of the algebras  $\text{Cl}(n)$  and  $\mathbb{C}\text{l}(n)$  due to the following table:*

	$\text{Cl}(n)$	$\mathbb{C}\text{l}(n)$
1	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{H}$	$M_2(\mathbb{C})$
3	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$
5	$M_4(\mathbb{C})$	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$

*Proof.* See theorem I.4.3 of [LM89].  $\square$

The algebras  $\text{Cl}(n)$  and  $\mathbb{C}\text{l}(n)$  for  $n \geq 3$  are shown to be isomorphic to  $M_k(\mathbb{F})$  or  $M_k(\mathbb{F}) \oplus M_k(\mathbb{F})$  (with  $\mathbb{F} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ) by using the following isomorphisms.

$$\begin{aligned} M_l(\mathbb{R}) \otimes_{\mathbb{R}} M_m(\mathbb{R}) &\cong M_{lm}(\mathbb{R}) \\ M_l(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{F} &\cong M_l(\mathbb{F}) \\ M_l(\mathbb{C}) \otimes_{\mathbb{C}} M_m(\mathbb{C}) &\cong M_{lm}(\mathbb{C}) \end{aligned}$$

The classification shows that  $\text{Cl}(n)$  splits into a sum of two matrix algebras exactly when  $n \equiv 3 \pmod{4}$  and similarly  $\mathbb{C}\text{l}(n)$  splits into a sum of two matrix algebras exactly when  $n$  is odd. This splitting can be understood by considering the volume element  $\omega$  given by

$$\omega = e_1 \cdots e_n$$

where  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $\mathbb{R}^n$ . Note that  $\omega$  is independent of the choice of basis once we have fixed an orientation of  $\mathbb{R}^n$  and  $e_1, \dots, e_n$  are

chosen to be positively oriented. For  $n$  odd, define the complex volume element  $\omega_{\mathbb{C}}$  by

$$\omega_{\mathbb{C}} = i^{\frac{n+1}{2}} e_1 \cdots e_n.$$

Suppose that  $n \equiv 3 \pmod{4}$ . Then

$$\omega^2 = (-1)^{(n-1)+(n-2)+\cdots+1} e_1^2 \cdots e_n^2 = (-1)^{\frac{n(n-1)}{2}} (-1)^n = (-1)^{\frac{n(n+1)}{2}} = 1$$

and

$$e_i \omega = (-1)^{i-1} e_1 \cdots e_i e_i \cdots e_n = (-1)^{i-1} (-1)^{n-i} e_1 \cdots e_n e_i = \omega e_i \quad \text{for } i \in \{1, \dots, n\}.$$

Hence, if  $\text{Cl}(n)^+$  and  $\text{Cl}(n)^-$  denote the  $+1$  and  $-1$  eigenspaces of  $\text{Cl}(n)$  under multiplication by  $\omega$ , then

$$\text{Cl}(n) = \text{Cl}(n)^+ \oplus \text{Cl}(n)^-$$

and  $\text{Cl}(n)^{\pm}$  are ideals since if  $x^{\pm} \in \text{Cl}(n)^{\pm}$  then

$$\omega e_i x^{\pm} = e_i \omega x^{\pm} = \pm e_i x^{\pm} \quad \text{for } i \in \{1, \dots, n\}.$$

Since  $\omega \in \text{Cl}(n)^1$ , we get that

$$\overline{\alpha}(\text{Cl}(n)^{\pm}) = \text{Cl}(n)^{\mp}$$

and so  $\text{Cl}(n)^+$  and  $\text{Cl}(n)^-$  are isomorphic. Similarly, if  $n$  is even,  $\omega_{\mathbb{C}}^2 = 1$  and  $\omega_{\mathbb{C}}$  lies in the center of  $\mathbb{C}\text{Cl}(n)$  and therefore we get a splitting into ideals

$$\mathbb{C}\text{Cl}(n) = \mathbb{C}\text{Cl}(n)^+ \oplus \mathbb{C}\text{Cl}(n)^-$$

where  $\mathbb{C}\text{Cl}(n)^{\pm}$  denote the  $\pm 1$  eigenspaces under multiplication by  $\omega_{\mathbb{C}}$ . These splittings correspond to the splittings given in theorem 2.3.

The classification in theorem 2.3 shows that the representation theory of  $\text{Cl}(n)$  and  $\mathbb{C}\text{Cl}(n)$  is particularly simple, since up to equivalence,  $M_k(\mathbb{F})$  (with  $\mathbb{F} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ) only has one irreducible representation over  $\mathbb{R}$ , namely the standard representation  $\rho: M_k(\mathbb{F}) \rightarrow \text{End}_{\mathbb{F}}(\mathbb{F}^k)$ , and similarly  $M_k(\mathbb{F}) \oplus M_k(\mathbb{F})$  only has two irreducible representations, namely  $\rho_1$  and  $\rho_2$  where for  $(\varphi_1, \varphi_2) \in M_k(\mathbb{F}) \oplus M_k(\mathbb{F})$

$$\rho_i(\varphi_1, \varphi_2) = \rho(\varphi_i) \quad \text{for } i = 1, 2.$$

Note that a representation over  $\mathbb{C}$  of a complex matrix algebra is any representation over  $\mathbb{R}$  which commutes with multiplication by  $i$ . We therefore get the following result:

**Proposition 2.4.** *When  $n \equiv 0, 1, 2 \pmod{4}$ , then up to equivalence there is exactly one irreducible real representation  $\sigma$  of  $\text{Cl}(n)$ .*

*When  $n \equiv 3 \pmod{4}$ , then up to equivalence there are exactly two irreducible real representations  $\sigma_1$  and  $\sigma_2$  of  $\text{Cl}(n)$  where if  $(\varphi_1, \varphi_2) \in \text{Cl}(n)^+ \oplus \text{Cl}(n)^-$  then*

$$\sigma_i(\varphi_1, \varphi_2) = \sigma(\varphi_i) \quad \text{for } i = 1, 2$$

where  $\sigma$  is an irreducible representation of  $\text{Cl}(n)^+ \cong \text{Cl}(n)^-$ .

When  $n$  is even, then up to equivalence there is exactly one irreducible complex representation  $\sigma$  of  $\text{Cl}(n)$ .

When  $n$  is odd, then up to equivalence there are exactly two irreducible complex representations  $\sigma_1$  and  $\sigma_2$  of  $\text{Cl}(n)$  where if  $(\varphi_1, \varphi_2) \in \text{Cl}(n)^+ \oplus \text{Cl}(n)^-$  then

$$\sigma_i(\varphi_1, \varphi_2) = \sigma(\varphi_i) \quad \text{for } i = 1, 2$$

where  $\sigma$  is an irreducible representation of  $\text{Cl}(n)^+ \cong \text{Cl}(n)^-$ .

Since  $\text{Cl}(n)$  is either of the form  $M_k(\mathbb{F})$  or a direct sum  $M_k(\mathbb{F}) \oplus M_k(\mathbb{F})$ , we can think of  $\text{Cl}(n)$  as a Lie algebra with Lie bracket given by

$$[x, y] = xy - yx \quad \text{for all } x, y \in \text{Cl}(n).$$

Let

$$\text{Cl}^*(n) = \{x \in \text{Cl}(n) \mid x \text{ is invertible}\}$$

be the multiplicative group of units in  $\text{Cl}(n)$ . Then  $\text{Cl}^*(n)$  will be a Lie group with Lie algebra  $\text{Cl}(n)$ . The usual exponential mapping  $\exp: \text{Cl}(n) \rightarrow \text{Cl}^*(n)$  given by

$$\exp(y) = \sum_{m=0}^{\infty} \frac{1}{m!} y^m$$

is well-defined and we get the adjoint representation  $\text{Ad}_{\text{Cl}}: \text{Cl}^*(n) \rightarrow \text{Aut}(\text{Cl}(n))$  with differential  $\text{ad}_{\text{Cl}}: \text{Cl}(n) \rightarrow \text{End}(\text{Cl}(n))$  where

$$\text{Ad}_{\text{Cl}}(g)x = gxg^{-1} \quad \text{for } g \in \text{Cl}^*(n), x \in \text{Cl}(n)$$

$$\text{ad}_{\text{Cl}}(x)(y) = [x, y] \quad \text{for } x, y \in \text{Cl}(n).$$

## 2.2 The spin group

Now we study the spin group which is a Lie group that lies in  $\text{Cl}(n)$ . We see that there is an important representation of the spin group called the spinor representation. We consider some properties of the spinor representation and find its weights.

Let

$$\text{Spin}(n) = \{v_1 \cdots v_{2k} \in \text{Cl}(n) \mid v_j \in V, q(v_j) \in \{\pm 1\}\}.$$

This is a compact subgroup of  $\text{Cl}^*(n)$  and is called the spin group. Let

$$\text{SO}(n) = \{x \in \text{GL}(\mathbb{R}^n) \mid q(xv) = q(v) \text{ for all } v \in \mathbb{R}^n, \det(x) = 1\}.$$

We then have the following important characterization of  $\text{Spin}(n)$ .

**Theorem 2.5.** *The map*

$$\psi: \text{Spin}(n) \rightarrow \text{SO}(n), \quad \psi(x)v = xv x^{-1}$$

*is a two-fold covering homomorphism.*

*Proof.* See theorem I.6.3 [GM91] or theorem I.2.9 [LM89].  $\square$

$\text{SO}(n)$  is a connected Lie group for  $n \geq 1$  and its fundamental group is  $\pi_1(\text{SO}(n)) = \mathbb{Z}_2$  for  $n \geq 3$  (see Knapp [Kna02] proposition 1.136). Hence  $\text{Spin}(n)$  is the universal covering group of  $\text{SO}(n)$  for  $n \geq 3$ . Since  $\text{SO}(2) = S^1$ , we have that  $\text{Spin}(2) = S^1$  and since  $\text{SO}(1) = \{1\}$ ,  $\text{Spin}(1) = \{\pm 1\}$ . We have the following result (see 3.24 Warner [War83]):

**Theorem 2.6.** *There is a unique differentiable structure on  $\text{Spin}(n)$  such that  $\psi: \text{Spin}(n) \rightarrow \text{SO}(n)$  is differentiable and non-singular. With this differentiable structure,  $\text{Spin}(n)$  becomes a Lie group and  $\psi$  a Lie group homomorphism.*

So when we think of  $\text{Spin}(n)$  as a Lie group in this way, proposition 3.26 of [War83] gives us that its Lie algebra is isomorphic to

$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X + X^t = 0\}.$$

Now since  $\text{Spin}(n)$  is also a closed subgroup of  $\text{Cl}^*(n)$ , it inherits a Lie group structure from  $\text{Cl}^*(n)$ . The following theorem shows that this is in fact the same.

**Proposition 2.7.** *As a Lie subgroup of  $\text{Cl}^*(n)$ ,  $\text{Spin}(n)$  has Lie algebra  $\mathfrak{spin}(n)$  given by the bivectors in  $\text{Cl}(n)$ :*

$$\mathfrak{spin}(n) = \text{span}_{\mathbb{R}}\{e_i e_j \mid i < j\} \subset \text{Cl}(n).$$

*With this differentiable structure,  $\psi: \text{Spin}(n) \rightarrow \text{SO}(n)$  is differentiable and non-singular. Furthermore,*

$$\text{ad}_{\text{Cl}}(\mathfrak{spin}(n)) \subset \text{End}(\mathbb{R}^n)$$

*and can in this way be thought of as  $\mathfrak{so}(n)$  (acting on  $\mathbb{R}^n$  as the differential of the standard representation of  $\text{SO}(n)$  on  $\mathbb{R}^n$ ). Thus the following diagram commutes:*

$$\begin{array}{ccccc} & & \text{Cl}(n) & \xrightarrow{\text{ad}_{\text{Cl}}} & \text{End}(\text{Cl}(n)) \\ & \swarrow \exp & \uparrow & & \uparrow \\ & & \mathfrak{spin}(n) & \xrightarrow[\cong]{\psi_*} & \mathfrak{so}(n) \\ & & \downarrow \exp & & \downarrow \exp \\ \text{Cl}^*(n) & \hookleftarrow \text{Spin}(n) & \xrightarrow{\psi} & \text{SO}(n) & \end{array}$$

*Proof.* See theorem I.8.10 of [GM91], lemma 1.1. of [Par72].  $\square$

Now we consider the representation theory of  $\text{Spin}(n)$ . Some obvious representations of  $\text{Spin}(n)$  are the ones induced by the map  $\psi: \text{Spin}(n) \rightarrow \text{SO}(n)$ . All such representations must be trivial on the element  $-1$ . However, using the fact that  $\text{Spin}(n) \subset \text{Cl}(n) \subset \mathbb{C}\text{l}(n)$ , we obtain a representation of  $\text{Spin}(n)$  which is non-trivial on  $-1$ . Proposition 2.4 showed that there are at most two inequivalent irreducible representations of  $\mathbb{C}\text{l}(n)$  over  $\mathbb{C}$ . Take any such representation and restrict it to  $\text{Spin}(n) \subset \text{Cl}(n)$ . We see (proposition 2.8 below) that up to equivalence, this gives exactly one representation of  $\text{Spin}(n)$  and this is called the (complex) spinor representation of  $\text{Spin}(n)$  which we denote by  $\sigma: \text{Spin}(n) \rightarrow \text{GL}_{\mathbb{C}}(S)$ . The elements of  $S$  are called spinors. We note that  $\sigma(-1) = -\text{Id}$ . Because of the classification in theorem 2.3, it is clear that  $S$  has complex dimension

$$\dim_{\mathbb{C}} S = 2^m \quad \text{for } n = 2m, n = 2m + 1.$$

**Proposition 2.8.** *Let  $\sigma: \text{Spin}(n) \rightarrow \text{GL}_{\mathbb{C}}(S)$  be the complex spinor representation of  $\text{Spin}(n)$ .*

*When  $n$  is odd,  $\sigma$  is independent of the choice of irreducible representation of  $\mathbb{C}\text{l}(n)$  used to define  $\sigma$  and furthermore  $\sigma$  is an irreducible representation of  $\text{Spin}(n)$ .*

*When  $n = 2m$  is even,  $\sigma$  splits into a direct sum of two inequivalent irreducible complex representations  $\sigma^{\pm}: \text{Spin}(n) \rightarrow \text{GL}_{\mathbb{C}}(S^{\pm})$  where  $\dim_{\mathbb{C}} S^{\pm} = 2^{m-1}$ .*

*Proof.* (Similar to that of proposition I.5.12 of [LM89]). We note that due to proposition 2.2,

$$\text{Spin}(n) \subset \mathbb{C}\text{l}(n)^0 \cong \mathbb{C}\text{l}(n-1),$$

and since  $\text{Spin}(n)$  contains an additive basis of  $\mathbb{C}\text{l}(n)^0$ , any irreducible representation of  $\mathbb{C}\text{l}(n)^0$  restricts to an irreducible representation of  $\text{Spin}(n)$ .

Suppose that  $n = 2m + 1$  is odd. We saw that in this case  $\mathbb{C}\text{l}(n) = \mathbb{C}\text{l}(n)^+ \oplus \mathbb{C}\text{l}(n)^-$  where  $\overline{\alpha}(\mathbb{C}\text{l}(n)^{\pm}) = \mathbb{C}\text{l}(n)^{\mp}$ . We must have that

$$\mathbb{C}\text{l}(n)^0 = \{(\varphi, \overline{\alpha}(\varphi)) \in \mathbb{C}\text{l}(n)^+ \oplus \mathbb{C}\text{l}(n)^- \mid \varphi \in \mathbb{C}\text{l}(n)^+\}$$

and therefore the two inequivalent irreducible representations of  $\mathbb{C}\text{l}(n)$  restricted to  $\mathbb{C}\text{l}(n)^0$  become  $\sigma_1$  and  $\sigma_2 = \sigma_1 \circ \overline{\alpha}$ . Hence, as representations of  $\mathbb{C}\text{l}(n)^0$ , they are equivalent. This gives a representation of  $\mathbb{C}\text{l}(n)^0 \cong \mathbb{C}\text{l}(2m) \cong \text{M}_{2^m}(\mathbb{C})$  of complex dimension  $2^m$  and therefore it must be irreducible.

Suppose that  $n$  is even. Let  $\omega_{\mathbb{C}} = i^{\frac{n}{2}} e_1 \cdots e_{n-1}$  be the volume element of  $\mathbb{C}\text{l}(n-1)$ . Then under the isomorphism  $\mathbb{C}\text{l}(n)^0 \cong \mathbb{C}\text{l}(n-1)$ ,  $\omega_{\mathbb{C}}$  is taken to the element

$$\begin{aligned} \omega'_{\mathbb{C}} &= i^{\frac{n}{2}} (e_1 e_n) \cdots (e_{n-1} e_n) = i^{\frac{n}{2}} (-1)^{1+\cdots+(n-2)} e_1 \cdots e_{n-1} e_n^{n-1} \\ &= i^{\frac{n}{2}} (-1)^{\frac{(n-1)(n-2)}{2} + \frac{n-2}{2}} e_1 \cdots e_n = i^{\frac{n}{2}} e_1 \cdots e_n \end{aligned}$$

and so  $(\omega'_\mathbb{C})^2 = 1$  and  $\omega'_\mathbb{C}$  commutes with the elements of  $\mathbb{C}l(n)^0$ . Hence if  $S^\pm$  denote the  $\pm 1$  eigenspaces of  $\sigma(\omega'_\mathbb{C})$ ,  $\sigma$  restricts to representations  $\sigma^\pm$  of  $\mathbb{C}l(n)^0$  on  $S^\pm$  that correspond to the two inequivalent representations of  $\mathbb{C}l(n-1)$ .  $\square$

The representations  $\sigma^\pm$  are called the half spinor representations of  $\text{Spin}(n)$ .

**Proposition 2.9.** *Let  $\sigma_*: \mathfrak{spin}(n) \rightarrow \text{End}(S)$  denote the differential of the spinor representation. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathfrak{spin}(n) & \xrightarrow{\sigma_*} & \text{End}(S) \\ \downarrow & & \uparrow \text{Id} \\ \mathbb{C}l(n) & \xrightarrow{\sigma} & \text{End}(S) \\ \uparrow & & \uparrow \\ \text{Spin}(n) & \xrightarrow{\sigma} & \text{GL}(S) \end{array}$$

where  $\text{Id}$  denotes the identification

$$v = \left. \frac{d}{dt} \right|_{t=0} \exp(tv) \quad \text{for } v \in \text{End}(S).$$

*Proof.* See corollary 1.1 [Par72].  $\square$

Assume that  $n = 2m$  or  $n = 2m + 1$ . We now find the weights of the complex spinor representation  $\sigma: \text{Spin}(n) \rightarrow \text{GL}_\mathbb{C}(S)$ . In order to do so we consider  $\sigma_*: \mathfrak{spin}(n) \rightarrow \text{End}_\mathbb{C}(S)$  which we can also denote by  $\sigma$  due to proposition 2.9. For each  $k \in \{1, \dots, m\}$  define

$$\omega_k = -ie_{2k-1}e_{2k}$$

where  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $\mathbb{R}^n$ . It is easy to see that

$$\omega_i \omega_j = \omega_j \omega_i \quad \text{for } i, j \in \{1, \dots, m\} \quad (2.1)$$

$$\omega_k^2 = 1 \quad \text{for } k \in \{1, \dots, m\} \quad (2.2)$$

$$\omega_k e_j = \begin{cases} -e_j \omega_k & \text{for } j = 2k, j = 2k - 1 \\ e_j \omega_k & \text{otherwise} \end{cases} \quad (2.3)$$

Hence if we define

$$\mathfrak{t} = \text{span}_\mathbb{R} \{i\omega_k \mid k = 1, \dots, m\},$$

then clearly  $\mathfrak{t}$  is an abelian Lie subalgebra of  $\mathfrak{spin}(n)$  of dimension  $m$ . We know that the Cartan subalgebra of  $\mathfrak{so}(n) \cong \mathfrak{spin}(n)$  has dimension  $m$  and therefore  $\mathfrak{t}$  is the Cartan subalgebra of  $\mathfrak{spin}(n)$ .

We now consider the action of the  $\omega_k$ 's on  $S$  under  $\sigma: \mathfrak{spin}(n) \rightarrow \text{End}(S)$  (where we have extended  $\sigma$  complex linearly to  $\mathfrak{spin}(n) \oplus i\mathfrak{spin}(n)$ ). Since  $\sigma$  is a homomorphism, (2.2) shows that  $S$  splits into a direct sum  $S = S_+ \oplus S_-$  where  $S_\pm$  are the  $\pm 1$  eigenspaces of  $\sigma(\omega_1)$ . Since

$$\sigma(e_j)^2 = \sigma(e_j^2) = -\text{Id} \quad \text{for } j \in \{1, \dots, n\},$$



$\sigma(e_j)$  is an isomorphism for all  $j$  and (2.3) gives us that

$$\sigma(e_1)(S_{\pm}) = S_{\mp}.$$

Hence

$$\dim S_+ = \dim S_- = \frac{1}{2} \dim S = 2^{m-1}.$$

Since  $\sigma(\omega_1)$  commutes with  $\sigma(\omega_k)$  for all  $k$ ,  $S_{\pm}$  are invariant under  $\sigma(\omega_k)$  for all  $k$ . We can therefore repeat this process with  $\omega_2$  in place of  $\omega_1$ ,  $S_{\pm}$  in place of  $S$  and  $e_3$  in place of  $e_1$ . This gives us a splitting

$$S = S_{++} \oplus S_{+-} \oplus S_{-+} \oplus S_{--}$$

where  $S_{\pm\pm}$  are simultaneous eigenspaces of  $\omega_1$  and  $\omega_2$ , each of dimension  $2^{m-2}$ . Continueing in this fashion gives us the splitting

$$S = \bigoplus_{\varepsilon \in \{\pm 1\}^m} S_{\varepsilon}$$

where for each  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{\pm 1\}^m$ ,  $S_{\varepsilon}$  is the simultaneous  $\varepsilon_k$  eigenspace of  $\omega_k$ . Note that  $\dim S_{\varepsilon} = 1$  for all  $\varepsilon$ .

For  $n = 2m$ , we saw in the proof of proposition 2.8 that  $S^{\pm}$  are the  $\pm 1$  eigenspaces of  $\sigma(\omega'_C)$  where

$$\omega'_C = i^m e_1 \cdots e_n = \omega_1 \cdots \omega_m.$$

For each  $s \in S_{\varepsilon}$

$$\sigma(\omega'_C)s = \varepsilon_1 \cdots \varepsilon_m s.$$

So if

$$\begin{aligned} E^+ &= \{\varepsilon \in \{\pm 1\}^m \mid \varepsilon_k = -1 \text{ for an even number of } \varepsilon_k\} \\ E^- &= \{\pm 1\}^m \setminus E^+, \end{aligned} \tag{2.4}$$

it is clear that

$$S^+ = \bigoplus_{\varepsilon \in E^+} S_{\varepsilon}, \quad S^- = \bigoplus_{\varepsilon \in E^-} S_{\varepsilon}.$$

This leads us to the following result.

**Proposition 2.10.** *The weights of the spinor representation  $\sigma: \text{Spin}(n) \rightarrow \text{GL}_{\mathbb{C}}(S)$  for  $n = 2m$ ,  $n = 2m + 1$  are given by*

$$\lambda_{\varepsilon} = \frac{1}{2} \sum_{k=1}^m \varepsilon_k \eta_k \circ \psi_*, \quad \text{for } \varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{\pm 1\}^m$$

where  $\{\eta_k \mid 1 \leq k \leq m\}$  are the weights of the standard representation of  $\text{SO}(n)$  on  $\mathbb{R}^n$ .

For  $n = 2m$  the weights of the representation  $\sigma^+$  are  $\{\lambda_{\varepsilon} \mid \varepsilon \in E^+\}$  and the weights of  $\sigma^-$  are  $\{\lambda_{\varepsilon} \mid \varepsilon \in E^-\}$ .

The multiplicity of each  $\lambda_{\varepsilon}$  is the number of ways in which  $\lambda_{\varepsilon}$  can be written in the above form.

*Proof.* Let  $\mathfrak{t}_{\mathbb{C}}$  be the complexification of  $\mathfrak{t}$  and consider the action of  $\mathfrak{t}_{\mathbb{C}}$  on  $S$  under  $\sigma$ . For  $x = \sum_{k=1}^m x_k \omega_k \in \mathfrak{t}_{\mathbb{C}}$  we have that

$$\sigma(x)(s) = \left( \sum_{k=1}^m x_k \varepsilon_k \right) s \quad \text{for } s \in S_{\varepsilon}.$$

Hence if  $\lambda_{\varepsilon} \in \mathfrak{t}_{\mathbb{C}}^*$  is defined by

$$\lambda_{\varepsilon} \left( \sum_{k=1}^m x_k \omega_k \right) = \sum_{k=1}^m \varepsilon_k x_k$$

then

$$\sigma(x)(s) = \lambda_{\varepsilon}(x)(s) \quad \text{for } s \in S_{\varepsilon}.$$

So for each  $\varepsilon \in \{\pm 1\}^m$ ,  $S_{\varepsilon}$  is a weight space of  $\sigma$  with weight  $\lambda_{\varepsilon}$ . It is clear that if  $n = 2m$ , then for  $\varepsilon \in E^{\pm}$ ,  $S_{\varepsilon}$  is a weight space of  $\sigma^{\pm}$  with weight  $\lambda_{\varepsilon}$ . Since  $\dim S_{\varepsilon} = 1$ , the multiplicity of each  $\lambda_{\varepsilon}$  is the number of ways in which  $\lambda_{\varepsilon}$  can be expressed in the above form.

Now proposition 2.7 showed that the differential of the standard representation of  $\mathrm{SO}(n)$  is the representation  $\mathrm{ad}_{\mathrm{Cl}} \circ \psi_*^{-1}: \mathfrak{so}(n) \rightarrow \mathrm{End}(\mathbb{R}^n)$  which we extend complex linearly to get  $\mathrm{ad}_{\mathrm{Cl}} \circ \psi_*^{-1}: \mathfrak{so}(n) \oplus i\mathfrak{so}(n) \rightarrow \mathrm{End}(\mathbb{C}^n)$ . We have that for  $j, k \in \{1, \dots, m\}$

$$\mathrm{ad}_{\mathrm{Cl}}(\omega_j)(e_{2k-1} \pm ie_{2k}) = \begin{cases} \mp 2(e_{2k-1} \pm ie_{2k}) & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}.$$

So if for  $k \in \{1, \dots, m\}$ ,  $\eta_k \in \mathfrak{t}_{\mathbb{C}}^*$  is defined as

$$\eta_k \circ \psi_* \left( \sum_{j=1}^m x_j \omega_j \right) = 2x_k,$$

then for all  $x \in \mathfrak{t}_{\mathbb{C}}$  and  $k \in \{1, \dots, m\}$  we have that

$$\mathrm{ad}_{\mathrm{Cl}}(x)v = \begin{cases} \eta_k(\psi_* x)v & \text{if } v \in \mathrm{span}\{e_{2k-1} - ie_{2k}\} \\ -\eta_k(\psi_* x)v & \text{if } v \in \mathrm{span}\{e_{2k-1} + ie_{2k}\} \end{cases}.$$

In the case  $n = 2m + 1$  we also have that

$$\mathrm{ad}_{\mathrm{Cl}}(\omega_j)(e_{2m+1}) = 0 \quad \text{for } j \in \{1, \dots, m\}$$

and therefore

$$\mathrm{ad}_{\mathrm{Cl}}(x)(v) = 0 \quad \text{for } x \in \mathfrak{t}_{\mathbb{C}}, v \in \mathrm{span}\{e_{2m+1}\}.$$

Since  $\{e_{2k-1} \pm ie_{2k} \mid 1 \leq k \leq m\}$  is a basis of  $\mathbb{C}^{2m}$  and  $\{e_{2k-1} \pm ie_{2k} \mid 1 \leq k \leq m\} \cup \{e_{2m+1}\}$  is a basis of  $\mathbb{C}^{2m+1}$ , we get that the weights of the standard representation of  $\mathrm{SO}(n)$  are  $\{\pm \eta_k \mid 1 \leq k \leq m\}$ . Since

$$\lambda_{\varepsilon} = \frac{1}{2} \sum_{k=1}^m \varepsilon_k \eta_k \circ \psi_*,$$

this completes the proof.  $\square$

### 3 Spin structure on $G/H$

#### 3.1 The tangent bundle on $G/H$

We now turn our attention to homogeneous spaces  $G/H$ . In section 3.2 we see that in some cases it is possible to define a so-called spin structure of the tangent bundle  $T(G/H)$  of  $G/H$ . This gives a vector bundle called the spinor bundle on  $G/H$  in which each fibre is isomorphic to the space of spinors  $S$ . The construction is described in [Par72] and [LM89].

Let  $G$  be a smooth connected real Lie group with Lie algebra  $\mathfrak{g}$ . We have the adjoint representation of  $G$  denoted by  $\text{Ad}: G \rightarrow \text{End}(\mathfrak{g})$ . Suppose that we have a closed connected subgroup  $H$  of  $G$  with Lie algebra  $\mathfrak{h}$  such that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  (as vector spaces) where  $\mathfrak{p}$  is invariant under  $\text{Ad}(H)$  and there is an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{p}$  such that  $(\mathfrak{p}, \text{Ad})$  is an orthogonal representation of  $H$ . Then  $G/H$  becomes a manifold such that the canonical projection  $\pi: G \rightarrow G/H$  is differentiable (see theorem 3.58 in [War83]) and the differential of  $\pi$  at the identity  $e \in G$  is a vector space isomorphism  $\pi_*: \mathfrak{p} \rightarrow T_H(G/H)$  and  $\pi_*(\mathfrak{h}) = 0$ .

We now recall the notion of an induced vector bundle. Let  $(V, \tau)$  be a representation of  $H$ . Let  $\underline{V} = G \times_H V$  denote the set of equivalence classes of  $G \times V$  under the equivalence relation

$$(g, v) \sim (gh, \tau(h^{-1})v) \quad \text{for } g \in G, v \in V, h \in H.$$

Then  $\underline{V}$  is a smooth vector bundle over  $G/H$  and we call it the induced vector bundle of  $(V, \tau)$  (see Kobayashi and Nomizu [KN96] p.54).  $G$  acts on the left of  $\underline{V}$  by

$$g'[g, v] = [g'g, v] \quad \text{for } g' \in G, [g, v] \in \underline{V}$$

where  $[g, v]$  denotes the equivalence class of  $(g, v) \in G \times V$ .

The tangent bundle  $T(G/H)$  of  $G/H$  can be identified with the induced vector bundle  $\underline{\mathfrak{p}} = G \times_H \mathfrak{p}$  of the representation  $(\text{Ad}, \mathfrak{p})$ . Let  $L_g: G/H \rightarrow G/H$  denote left translation by  $g \in G$ , i.e.,  $L_g(g'H) = gg'H$  for  $g' \in G$ . For each  $g \in G$ , let  $(L_g)_*: T_H(G/H) \rightarrow T_{gH}(G/H)$  denote the differential of  $L_g$  at the identity  $H \in G/H$ . Then  $L_*: h \mapsto (L_h)_*$  is a representation of  $H$  on  $T_H(G/H)$  and under the identification  $\mathfrak{p} \cong T_H(G/H)$ , it is identified with  $(\text{Ad}, \mathfrak{p})$ , i.e., the following diagram commutes for all  $h \in H$ .

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\pi_*} & T_H(G/H) \\ \text{Ad}(h) \downarrow & & \downarrow (L_h)_* \\ \mathfrak{p} & \xrightarrow{\pi_*} & T_H(G/H) \end{array} \quad (3.1)$$

If  $X \in T_{gH}(G/H)$  then  $X = (L_g)_*(\pi_*(\xi))$  for some  $\xi \in \mathfrak{p}$  and we therefore get a map  $G \times_H \mathfrak{p} \rightarrow T_{gH}(G/H)$  where  $[g, \xi] \mapsto (L_g)_*(\pi_*(\xi))$ . This map is well-defined since (3.1) shows that for  $h \in H$

$$[gh, \text{Ad}(h^{-1})\xi] \mapsto (L_{gh})_*(\pi_*(\text{Ad}(h^{-1})\xi)) = (L_{gh})_*(L_{h^{-1}})_*(\pi_*(\xi)) = (L_g)_*(\pi_*(\xi)).$$

Similarly, the complexified tangent bundle can be identified with the induced bundle  $\underline{\mathfrak{p}}_{\mathbb{C}}$  (where the induced bundle is obtained by extending  $\text{Ad}$  complex linearly to  $\mathfrak{p}_{\mathbb{C}}$ ).

We extend the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{p}$   $G$ -invariantly to  $T(G/H)$ , i.e.,

$$\langle (L_g)_* X, (L_g)_* Y \rangle = \langle X, Y \rangle.$$

for  $X, Y \in T_H(G/H) \cong \mathfrak{p}$ . This gives a  $G$ -invariant Riemannian metric on  $T(G/H)$ . Similarly, the complexification of  $\langle \cdot, \cdot \rangle$  gives a hermitian inner product on  $\mathfrak{p}_{\mathbb{C}}$  and this gives the complexified Riemannian metric on the complexified tangent bundle  $T(G/H)_{\mathbb{C}}$ .

### 3.2 The spinor bundle on $G/H$

Now let  $\text{SO}(\mathfrak{p})$  denote the space

$$\text{SO}(\mathfrak{p}) = \{\varphi \in \text{End}(\mathfrak{p}) \mid \langle \varphi \xi, \varphi \xi \rangle = \langle \xi, \xi \rangle \text{ for all } \xi \in \mathfrak{p}, \det \varphi = 1\}.$$

Note that since  $H$  is connected and  $\text{Ad}$  continuous, we must have that  $\text{Ad}(H) \subset \text{SO}(\mathfrak{p})$ . Let  $\underline{\text{SO}}(\mathfrak{p}) = G \times_H \text{SO}(\mathfrak{p})$  be the fibre bundle induced by the representation in the group of diffeomorphisms given by

$$H \xrightarrow{\text{Ad}} \text{SO}(\mathfrak{p}) \xrightarrow{l} \text{Diff}(\text{SO}(\mathfrak{p}))$$

where  $l$  denotes left multiplication in  $\text{SO}(\mathfrak{p})$ . Then we can think of  $\underline{\text{SO}}(\mathfrak{p})$  as the principal  $\text{SO}(n)$  bundle of oriented orthogonal frames of  $T(G/H)$  where  $n = \dim \mathfrak{p}$ .

A spin structure of  $T(G/H)$  is a principal  $\text{Spin}(n)$  bundle  $P \rightarrow G/H$  such that there exists a two fold covering bundle map

$$\Psi: P \rightarrow \underline{\text{SO}}(\mathfrak{p})$$

such that if  $\psi: \text{Spin}(n) \rightarrow \text{SO}(n)$  is the two fold covering homomorphism of  $\text{SO}(n)$  then

$$\Psi(px) = \Psi(p)\psi(x) \quad \text{for } p \in P, x \in \text{Spin}(n). \quad (3.2)$$

We now construct a specific spin structure of  $T(G/H)$ . Let  $\text{Spin}(\mathfrak{p})$  denote the two-fold cover  $\text{SO}(\mathfrak{p})$  with covering map  $\psi$ . Suppose that the adjoint representation  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  lifts to  $\text{Spin}(\mathfrak{p})$ , i.e., there exists a homomorphism  $\bar{\rho}: H \rightarrow \text{Spin}(\mathfrak{p})$  such that the following diagram commutes

$$\begin{array}{ccc} & \text{Spin}(\mathfrak{p}) & \\ \bar{\rho} \nearrow & \downarrow \psi & \\ H & \xrightarrow{\text{Ad}} & \text{SO}(\mathfrak{p}). \end{array}$$

Note that if  $\bar{\rho}$  exists, it is unique since if there exists another homomorphism  $\bar{\rho}'$  with this property, then for all  $h \in H$

$$\psi(\bar{\rho}(h)\bar{\rho}'(h^{-1})) = \psi(\bar{\rho}(h))\psi(\bar{\rho}'(h^{-1})) = \text{Ad}(h)\text{Ad}(h^{-1}) = \text{Id}$$

and therefore

$$\bar{\rho}(h)\bar{\rho}'(h^{-1}) \in \{\pm 1\} \quad \text{for all } h \in H.$$

Since  $h \mapsto \bar{\rho}(h)\bar{\rho}'(h^{-1})$  is continuous on  $H$  and  $H$  is connected, we must have that  $h \mapsto \bar{\rho}(h)\bar{\rho}'(h^{-1})$  is constant. Since  $\bar{\rho}(e)\bar{\rho}'(e^{-1}) = 1$ , we conclude that  $\bar{\rho}' = \bar{\rho}$ .

The fibre bundle  $\underline{\text{Spin}}(\mathfrak{p}) = G \times_H \text{Spin}(\mathfrak{p})$  induced by the representation  $l \circ \bar{\rho}$ , where  $l$  denotes left multiplication in  $\text{Spin}(\mathfrak{p})$ , is a principal  $\text{Spin}(n)$  bundle. The covering homomorphism  $\psi: \text{Spin}(n) \rightarrow \text{SO}(n)$  induces a bundle map  $\Psi: \underline{\text{Spin}}(\mathfrak{p}) \rightarrow \underline{\text{SO}}(\mathfrak{p})$  given by

$$\Psi[g, a] = [g, \psi(a)] \quad \text{for } [g, a] \in \underline{\text{Spin}}(\mathfrak{p}).$$

This is well-defined since

$$\Psi[gh, \bar{\rho}(h^{-1})a] = [gh, \psi(\bar{\rho}(h^{-1})a)] = [gh, \text{Ad}(h^{-1})\psi(a)] = [g, \psi(a)] \quad \text{for } h \in H$$

and clearly  $\Psi$  satisfies the condition (3.2). Hence,  $\underline{\text{Spin}}(\mathfrak{p})$  is a spin structure of  $T(G/H)$ . The spin structure is  $G$ -invariant in the sense that  $\Psi$  commutes with the action of  $G$ .

Recall that we have the spinor representation  $\sigma: \text{Spin}(n) \rightarrow \text{GL}(S)$ . When the adjoint representation  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  lifts to  $\text{Spin}(\mathfrak{p})$ , we get the spinor representation

$$\chi = \sigma \circ \bar{\rho} \tag{3.3}$$

of  $H$  and when  $n$  is even, we also have the half spinor representations

$$\chi^\pm = \sigma^\pm \circ \bar{\rho}.$$

Now we define the complex spinor bundle  $\underline{S}$  to be the vector bundle induced on  $G/H$  by  $(S, \chi)$ , i.e.,  $\underline{S} = G \times_H S$ . Note that this is equivalent to the induced bundle  $\underline{\text{Spin}}(\mathfrak{p}) \times_H S$  on the principal bundle  $\underline{\text{Spin}}(\mathfrak{p})$  under the representation  $(S, \sigma)$ . Similarly, if  $n$  is even, we get the half spinor bundles  $\underline{S}^\pm$  induced by  $(S^\pm, \chi^\pm)$  and in this case we have that

$$\underline{S} = \underline{S}^+ \oplus \underline{S}^-.$$

## 4 The Dirac operator

### 4.1 Sections of induced bundles

If  $G/H$  has a spin structure, it is possible to define the Dirac operator on  $G/H$ . This is an operator on the sections  $\Gamma(\underline{S} \otimes V)$  of an induced vector bundle  $\underline{S} \otimes V$

where  $(S, \chi)$  is the spinor representation of  $H$  and  $(V, \tau)$  is any complex representation of  $H$ . The Dirac operator is the composition of reductive covariant differentiation which we describe in section 4.2 and Clifford multiplication which we describe in section 4.3. First we study the sections of an induced bundle in more detail.

Let  $(V, \tau)$  be a representation of  $H$  over  $\mathbb{F}$  (where  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ) and let  $\underline{V}$  be the induced vector bundle on  $G/H$ . If  $\varphi \in \Gamma(\underline{V})$  is a section of  $\underline{V}$ , then it is of the form

$$\varphi(gH) = [g, \hat{\varphi}(g)] \quad \text{for } g \in G$$

where  $\hat{\varphi}: G \rightarrow V$  is a smooth map satisfying

$$\hat{\varphi}(gh) = \tau(h^{-1})\hat{\varphi}(g) \quad \text{for } g \in G, h \in H.$$

Hence we can make the identification

$$\Gamma(\underline{V}) \cong \{\hat{\varphi}: G \rightarrow V \mid \hat{\varphi} \text{ smooth, } \hat{\varphi}(gh) = \tau(h^{-1})\hat{\varphi}(g) \text{ for } g \in G, h \in H\}.$$

Identifying the smooth maps  $G \rightarrow V$  with  $C^\infty(G) \otimes V$  where  $C^\infty(G)$  denotes the smooth maps  $G \rightarrow \mathbb{F}$ , we therefore get the identification

$$\Gamma(\underline{V}) \cong \{\hat{\varphi} \in C^\infty(G) \otimes V \mid (\tilde{r} \otimes \tau)(h)\hat{\varphi} = \hat{\varphi} \text{ for } h \in H\} \quad (4.1)$$

where  $\tilde{r}$  denotes the right regular action of  $G$  on  $C^\infty(G)$ , i.e.,

$$\tilde{r}(g)f(g') = f(g'g) \quad \text{for } g, g' \in G, f \in C^\infty(G).$$

In this identification, an element  $\hat{\varphi}$  belonging to the right hand side of (4.1) is of the form

$$\hat{\varphi} = \sum_i \tilde{\varphi}_i \otimes v_i$$

where  $\{v_i\}$  is a basis of  $V$ . By abuse of notation we write  $\tilde{\varphi} \otimes v$  to denote such an element and a section  $\varphi$  is identified with the element  $\hat{\varphi} = \tilde{\varphi} \otimes v$ .

We now see that  $\mathfrak{g}$  acts on  $\Gamma(\underline{V})$ .  $G$  acts on  $\Gamma(\underline{V})$  by the left regular action  $\tilde{l}$  given by

$$\tilde{l}(g)\varphi(g'H) = g \cdot \varphi(g^{-1}g'H) \quad \text{for } g, g' \in G, \varphi \in \Gamma(\underline{V}).$$

We have that

$$\widehat{\tilde{l}(g)\varphi(g')} = \hat{\varphi}(g^{-1}g')$$

so in terms of the isomorphism (4.1), the action is given by

$$\tilde{l}(g)(\tilde{\varphi} \otimes v)(g') = \tilde{\varphi}(g^{-1}g') \otimes v.$$

The differential of  $\tilde{l}$  gives an action  $\tilde{l}_*$  of  $\mathfrak{g}$  on  $\Gamma(\underline{V})$  where

$$\tilde{l}_*(\xi)(\tilde{\varphi} \otimes v)(g') = \frac{d}{dt}\bigg|_{t=0} \tilde{\varphi}(\exp(-t\xi)g') \otimes v = (\tilde{\xi}\tilde{\varphi} \otimes v)(g')$$

when  $[g', \xi] = (L_{g'})_*\xi \in T_{g'H}(G/H)$  for  $\xi \in \mathfrak{p}$  and  $\tilde{\xi}$  denotes the right invariant vector field on  $G$  induced by  $\xi$ .

## 4.2 Connections on induced bundles

In this section we define a way of differentiating sections of induced vector bundles in terms of a vector field. This is called covariant differentiation. In order to do so, we need the concept of a connection. We start by considering connections and covariant differentiation in general terms. We then study a specific connection called the reductive connection.

Let  $l_{g'}$ , respectively  $r_{g'}$  denote left and right translation by  $g'$  in  $G$ . A vector  $X \in T_g(G)$  is called vertical if it is tangent to the fibre  $\pi^{-1}(gH)$ , i.e.,  $\pi_*(l_{g^{-1}})_*X = 0$ . We denote the space of vertical vectors in  $T_g(G)$  by  $V_g$ . A connection on the tangent bundle  $T(G)$  of  $G$  is a choice of subspace  $Q_g \subset T_g(G)$  for each  $g \in G$  such that

- (i)  $T_g(G) = V_g \oplus Q_g$
- (ii)  $Q_{gh} = (r_h)_*(Q_g)$
- (iii)  $Q_g$  depends differentiably on  $g$ .

Suppose we have a connection on  $T(G)$ . For each  $g \in G$  the vectors of  $Q_g$  are called horizontal. If  $\gamma(t), t \in [0, t_0]$  is a (smooth) curve in  $G/H$ , a curve  $\tilde{\gamma}$  is called a horizontal lift of  $\gamma$  if  $\pi(\tilde{\gamma}(t)) = \gamma(t)$  for all  $t$  and the tangent vectors of  $\tilde{\gamma}$  are all horizontal. Given a curve  $\gamma$  in  $G/H$  and an element  $g \in \pi^{-1}(\gamma(0))$ , there is a unique horizontal lift  $\tilde{\gamma}$  of  $\gamma$  such that  $\tilde{\gamma}(0) = g$ . Hence for each curve  $\gamma$  in  $G/H$ , we have a well-defined map  $p(\gamma)_0^{t_0}: \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(t_0))$  taking a point  $g \in \pi^{-1}(\gamma(0))$  to  $\tilde{\gamma}(t_0)$  where  $\tilde{\gamma}$  is the horizontal lift of  $\gamma$  with starting point  $g$ . It can be shown that  $p(\gamma)_0^{t_0}$  is a linear isomorphism which is independent of the parametrization of  $\gamma$ . It is called the parallel displacement along  $\gamma$  from  $\gamma(0)$  to  $\gamma(t_0)$ . The inverse of  $p(\gamma)_0^{t_0}$  is parallel displacement along  $\gamma^{-1}$ . (See [KN96] section II.3).

Suppose that  $\underline{V}$  is an induced real vector bundle on  $G/H$  and that  $\gamma$  is a curve in  $G/H$ . Given an element  $[g, v] \in \underline{V}$ , the horizontal lift of  $\gamma$  to  $\underline{V}$  with starting point  $[g, v]$  is  $[\tilde{\gamma}, v]$  where  $\tilde{\gamma}$  is the horizontal lift of  $\gamma$  with starting point  $g$ . As before, we get the parallel displacement of fibres along  $\gamma$  from  $\gamma(0)$  to  $\gamma(t_0)$  given by

$$p(\gamma)_0^{t_0}[g, v] = [p(\gamma)_0^{t_0}g, v].$$

We now define covariant differentiation. Let  $\varphi \in \Gamma(\underline{V})$  and  $X \in T_{gH}(G/H)$ . The covariant derivative of  $\varphi$  in the direction of  $X$  is denoted  $\nabla_X \varphi$  and is defined as follows. Let  $\gamma(t), t \in [-t_0, t_0]$  be a curve in  $G/H$  with tangent vector  $X$  at  $\gamma(0) = gH$ . Then

$$\nabla_X \varphi = \lim_{s \rightarrow 0} \frac{(p(\gamma)_0^s)^{-1}(\varphi(\gamma(s))) - \varphi(gH)}{s}.$$

This is independent of the choice of  $\gamma$ . Now if  $X$  is a vector field on  $G/H$  we define the covariant derivative of  $\varphi$  with respect to  $X$  to be the element

$\nabla_X \varphi \in \Gamma(\underline{V})$  given by

$$\nabla_X \varphi(gH) = \nabla_{X_{gH}} \varphi \quad \text{for } g \in G.$$

This gives a map  $\nabla: \Gamma(\underline{V}) \rightarrow \Gamma(T^*(G/H) \otimes \underline{V})$  which we call the covariant derivative. We have the following result (see [KN96] proposition III.1.2).

**Proposition 4.1.** *The covariant derivative  $\nabla: \Gamma(\underline{V}) \rightarrow \Gamma(T^*(G/H) \otimes \underline{V})$  is linear and satisfies the Leibnitz rule, i.e., if  $f \in C^\infty(G/H)$  and  $\varphi \in \Gamma(\underline{V})$  then*

$$\nabla(f\varphi) = df \otimes \nabla\varphi + f\nabla\varphi.$$

In our case the splitting  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  gives a connection on  $T(G)$  by defining  $Q_g = (l_g)_* \mathfrak{p}$  for all  $g \in G$ . We call this the reductive connection. Now we consider the covariant derivative given by this connection. If  $X = (L_g)_* \xi \in T_{gH}(G/H)$  where  $\xi \in \mathfrak{p}$  (where we have identified  $\mathfrak{p}$  with  $T_H(G/H)$ ), then

$$\gamma(t) = g \exp(t\xi)H, \quad \text{for } t \in [-t_0, t_0]$$

defines a curve in  $G/H$  with tangent  $X$  at  $\gamma(0) = gH$ . In order to find the covariant derivative of a section  $\varphi \in \Gamma(\underline{V})$  in the direction of  $X$  we will need to compute

$$(p(\gamma)_0^s)^{-1}(\varphi(\gamma(s)))$$

for  $s \in [0, t_0]$ . For  $h \in H$ , the unique horizontal lift  $\tilde{\gamma}_h$  of  $\gamma$  with  $\tilde{\gamma}_h(0) = gh$  is given by

$$\tilde{\gamma}_h(t) = gh \exp(t \operatorname{Ad}(h^{-1})\xi) = g \exp(t\xi)h.$$

It is clear that  $\pi \circ \tilde{\gamma}_h(t) = \gamma(t)$  and  $\tilde{\gamma}$  is horizontal because of the  $\operatorname{Ad} H$ -invariance of  $\mathfrak{p}$ . Writing  $\varphi(gH)$  as  $[g, \hat{\varphi}(g)]$  we have that

$$\varphi(\gamma(s)) = [g \exp(s\xi), \hat{\varphi}(g \exp(s\xi))] = [\tilde{\gamma}_e(s), \hat{\varphi}(g \exp(s\xi))].$$

Hence

$$(p(\gamma)_0^s)^{-1}(\varphi(\gamma(s))) = [g, \hat{\varphi}(g \exp(s\xi))].$$

Note that  $(p(\gamma)_0^s)^{-1}(\varphi(\gamma(s)))$  is well-defined since if we chose a different representative of  $\varphi(\gamma(s))$ , then

$$\varphi(\gamma(s)) = [g \exp(s\xi)h, \tau^{-1}(h)\hat{\varphi}(g \exp(s\xi)h)] = [\tilde{\gamma}_h(s), \tau^{-1}(h)\hat{\varphi}(g \exp(s\xi)h)]$$

and

$$\begin{aligned} (p(\gamma)_0^s)^{-1}(\varphi(\gamma(s))) &= (p(\gamma)_0^s)^{-1}[g \exp(s\xi)h, \tau^{-1}(h)\hat{\varphi}(g \exp(s\xi)h)] \\ &= [gh, \tau^{-1}(h)\hat{\varphi}(g \exp(s\xi)h)] = [g, \hat{\varphi}(g \exp(s\xi))]. \end{aligned}$$

We conclude that

$$\nabla_X \varphi = \lim_{s \rightarrow 0} \frac{[g, \hat{\varphi}(g \exp(s\xi))] - [g, \hat{\varphi}(g)]}{s} = [g, \frac{d}{ds}|_{s=0} \hat{\varphi}(g \exp(s\xi))].$$



Now we describe  $\nabla$  in terms of the isomorphism (4.1). If  $\tilde{\varphi} \otimes v \in C^\infty(G) \otimes V$ , i.e.,  $\tilde{\varphi}(g) \otimes v = \tilde{\varphi}(g)$  for all  $g \in G$  we have that

$$\frac{d}{dt}|_{t=0} \tilde{\varphi}(g \exp(t\xi)) = (\tilde{r}_*(\xi) \otimes 1)(\tilde{\varphi} \otimes v)(g)$$

where  $\tilde{r}_*(\xi)$  denotes the differential of the right regular action of  $G$  on  $C^\infty(G)$ . Hence

$$\nabla_{[g, \xi]}(\tilde{\varphi} \otimes v)(g) = (\tilde{r}_*(\xi) \otimes 1)(\tilde{\varphi} \otimes v)(g).$$

Now let  $\{\xi_i\}$  be an orthonormal basis of  $\mathfrak{p}$  and let  $\{\xi_i^*\}$  be its dual basis, i.e.,

$$\xi_i^*(\xi) = \langle \xi_i, \xi \rangle \quad \text{for } \xi \in \mathfrak{p}.$$

The map  $\xi_i \mapsto \xi_i^*$  gives an isomorphism of  $\mathfrak{p}$  to  $\mathfrak{p}^*$ . Using the fact that  $T(G/H) \cong \mathfrak{p}$ , a vector field  $X$  on  $G/H$  can be thought of as an element  $\tilde{X} \otimes \xi \in C^\infty(G) \otimes \mathfrak{p}$  such that  $\tilde{X}(g) \otimes \xi = \tilde{X}(g)$  where  $[g, \tilde{X}(g)] = X(gH)$ . Let  $\tilde{X}(g) = \xi \in \mathfrak{p}$ .

We get that

$$\begin{aligned} \nabla_X(\tilde{\varphi} \otimes v)(g) &= \tilde{r}_*(\xi)\tilde{\varphi}(g) \otimes v = \sum_i \tilde{r}_*(\xi_i^*(\xi))\tilde{\varphi}(g) \otimes v \\ &= \sum_i (\tilde{r}_*(\xi_i) \otimes \xi_i^* \otimes 1)(\tilde{\varphi} \otimes \xi \otimes v)(g). \end{aligned}$$

So we may think of  $\nabla: \Gamma(\underline{V}) \rightarrow \Gamma(\underline{V} \otimes T^*(G/H)) \cong \Gamma(\underline{V} \otimes \mathfrak{p})$  as

$$\nabla(\tilde{\varphi} \otimes v) = \sum_i ((\tilde{r}_*(\xi_i) \otimes 1)(\tilde{\varphi} \otimes v)) \otimes \xi_i^* \cong \sum_i ((\tilde{r}_*(\xi_i) \otimes 1)(\tilde{\varphi} \otimes v)) \otimes \xi_i \quad (4.2)$$

when we think of sections in terms of (4.1).

Similarly, when  $\underline{V}$  is an induced complex vector bundle over  $G/H$ , we get the reductive covariant derivative  $\nabla: \Gamma(\underline{V}) \rightarrow \Gamma(\underline{V} \otimes (T^*(G/H))_{\mathbb{C}}) \cong \Gamma(\underline{V} \otimes \mathfrak{p}_{\mathbb{C}})$  where  $(T^*(G/H))_{\mathbb{C}}$  denotes the complexification of the cotangent bundle of  $G/H$  and in terms of the isomorphism (4.1),  $\nabla$  is given by (4.2).

### 4.3 The Dirac operator on $G/H$

Let  $(V, \tau)$  be a complex representation of  $H$ . We then have the representation  $(S \otimes V, \chi \otimes \tau)$  of  $H$  (see (3.3)), and we can form the induced bundle  $\underline{S \otimes V}$  which has the reductive covariant derivative  $\nabla$ . In this section we define Clifford multiplication and this enables us to define the Dirac operator. In proposition 4.2 we see that the Dirac operator has a simple expression in terms of the isomorphism (4.1). Finally, we look at some properties of the Dirac operator.

Recall that the complex spinor representation  $\sigma: \text{Spin}(\mathfrak{p}) \rightarrow \text{GL}(S)$  is the restriction of a representation  $\sigma: \mathbb{C}l(\mathfrak{p}) \rightarrow \text{End}(S)$ . Since  $\mathfrak{p}_{\mathbb{C}} \subset \mathbb{C}l(\mathfrak{p})$  we therefore get a map  $c: S \otimes \mathfrak{p}_{\mathbb{C}} \rightarrow S$  defined by

$$c(s \otimes \xi) = \sigma(\xi)s \quad \text{for } \xi \in \mathfrak{p}_{\mathbb{C}}, s \in S.$$

We call this Clifford multiplication. Note that for  $n = \dim \mathfrak{p}$  even, Clifford multiplication maps  $S^\pm \otimes \mathfrak{p}_\mathbb{C}$  into  $S^\mp$  respectively: Recall that  $S^\pm$  are the  $\pm 1$  eigenspaces of  $\sigma(\omega'_\mathbb{C})$  where  $\omega'_\mathbb{C} = i^{\frac{n}{2}} e_1 \cdots e_n$  for an oriented orthonormal basis  $\{e_j\}_{1 \leq j \leq n}$  of  $\mathfrak{p}$ . Since  $n$  is even,

$$\omega'_\mathbb{C} e_j = i^{\frac{n}{2}} e_1 \cdots e_n e_j = (-1)^{n-1} i^{\frac{n}{2}} e_j e_1 \cdots e_n = -e_j \omega'_\mathbb{C} \quad \text{for } j \in \{1, \dots, n\}$$

and therefore for  $s^\pm \in S^\pm$  and  $\xi \in \mathfrak{p}_\mathbb{C}$

$$\begin{aligned} \sigma(\omega'_\mathbb{C})(c(s^\pm \otimes \xi)) &= \sigma(\omega'_\mathbb{C})\sigma(\xi)s^\pm = -\sigma(\xi)\sigma(\omega'_\mathbb{C})s^\pm \\ &= \mp \sigma(\xi)s^\pm = \mp c(s^\pm \otimes \xi). \end{aligned}$$

Hence we get the maps  $c^\pm: S^\pm \otimes \mathfrak{p}_\mathbb{C} \rightarrow S^\mp$  by restricting  $c$ .

Now construct the induced bundle  $\underline{S \otimes \mathfrak{p}_\mathbb{C}}$  by the representation  $(S \otimes \mathfrak{p}_\mathbb{C}, \chi \otimes \text{Ad})$ . Clifford multiplication then induces a bundle map  $c: \underline{S \otimes \mathfrak{p}_\mathbb{C}} \rightarrow \underline{S}$  given by

$$c[g, s \otimes \xi] = [g, \sigma(\xi)s] \quad \text{for } s \in S, \xi \in \mathfrak{p}.$$

This is well-defined. To see this, observe that since  $\chi = \sigma \circ \bar{\rho}$ , we have that for  $h \in H$

$$\sigma(\text{Ad}(h^{-1})\xi)\chi(h^{-1})s = \sigma(\text{Ad}(h^{-1})(\xi)\bar{\rho}(h^{-1}))s.$$

Now since

$$\text{Ad}(h^{-1})\xi = \psi \circ \bar{\rho}(h^{-1})(\xi) = \bar{\rho}(h^{-1})\xi\bar{\rho}(h)$$

we get that

$$\sigma(\text{Ad}(h^{-1})\xi)\chi(h^{-1})s = \sigma(\bar{\rho}(h^{-1})\xi)s = \chi(h^{-1})\sigma(\xi)s.$$

Hence,

$$c[gh, \text{Ad}(h^{-1})\xi \otimes \chi(h^{-1})s] = [gh, \chi(h^{-1})\sigma(\xi)s] = c[g, \xi \otimes s].$$

Similarly when  $n$  is even, the maps  $c^\pm: S^\pm \otimes \mathfrak{p}_\mathbb{C} \rightarrow S^\mp$  give bundle maps  $c^\pm: \underline{S^\pm \otimes \mathfrak{p}_\mathbb{C}} \rightarrow \underline{S^\mp}$  on the bundles induced by the representations  $(S^\pm \otimes \mathfrak{p}_\mathbb{C}, \chi^\pm \otimes \text{Ad})$  and  $(S^\mp, \chi^\mp)$ .

Clifford multiplication  $c: \underline{S \otimes \mathfrak{p}_\mathbb{C}} \rightarrow \underline{S}$  induces a map  $c: \Gamma(\underline{S \otimes V \otimes \mathfrak{p}_\mathbb{C}}) \rightarrow \Gamma(\underline{S \otimes V})$ , which we also call Clifford multiplication, given by

$$c(\tilde{\varphi} \otimes s \otimes v \otimes \xi)(g) = \tilde{\varphi}(g) \otimes \sigma(\xi)s \otimes v$$

where  $\tilde{\varphi}(g) \otimes s \otimes v \otimes \xi = \hat{\varphi}(g)$  and  $\varphi(gH) = [g, \hat{\varphi}(g)]$ .

The Dirac operator is the operator  $D: \Gamma(\underline{S \otimes V}) \rightarrow \Gamma(\underline{S \otimes V})$  given by

$$D = c \circ \nabla.$$

Similarly, if  $n = \dim \mathfrak{p}$  is even we have the reductive covariant derivatives  $\nabla^\pm$  on  $\underline{S^\pm \otimes V}$  and Clifford multiplications  $c^\pm: \Gamma(\underline{S^\pm \otimes V \otimes \mathfrak{p}_\mathbb{C}}) \rightarrow \Gamma(\underline{S^\mp \otimes V})$  and we define the operators  $D^\pm: \Gamma(\underline{S^\pm \otimes V}) \rightarrow \Gamma(\underline{S^\mp \otimes V})$  by

$$D^\pm = c^\pm \circ \nabla^\pm.$$

Considering  $\Gamma(\underline{S \otimes V})$  as in (4.1), we see that  $D$  has a simple expression.

**Proposition 4.2.** *The Dirac operator  $D: \Gamma(\underline{S \otimes V}) \rightarrow \Gamma(\underline{S \otimes V})$  is in terms of (4.1) given by*

$$D = \sum_i \tilde{r}_*(\xi_i) \otimes \sigma(\xi_i) \otimes 1$$

where  $\{\xi_i\}$  is an orthonormal basis of  $\mathfrak{p}$  and  $\tilde{r}_*$  denotes the right regular action of  $\mathfrak{g}$  on  $C^\infty(G)$ .

*Proof.* We have that

$$\begin{aligned} D(\tilde{\varphi} \otimes s \otimes v) &= c \left( \sum_i (\tilde{r}_*(\xi_i) \otimes 1) (\tilde{\varphi} \otimes (s \otimes v)) \otimes \xi_i \right) \\ &= c \left( \sum_i \tilde{r}_*(\xi_i) \tilde{\varphi} \otimes s \otimes v \otimes \xi_i \right) \\ &= \sum_i \tilde{r}_*(\xi_i) \tilde{\varphi} \otimes \sigma(\xi_i) s \otimes v \\ &= \left( \sum_i \tilde{r}_*(\xi_i) \otimes \sigma(\xi_i) \otimes 1 \right) (\tilde{\varphi} \otimes s \otimes v). \end{aligned}$$

This completes the proof.  $\square$

We see that  $D$  is a first order differential operator which is homogeneous (i.e., it commutes with the action  $\tilde{l}$  of  $G$  on  $\Gamma(\underline{S \otimes V})$ ). The symbol  $\sigma_D$  of  $D$  is as follows. For  $\xi \in \mathfrak{p}$ ,  $\sigma_D(\xi): S \otimes V \rightarrow S \otimes V$  is the linear map given by

$$\sigma_D(\xi)(s \otimes v) = (\sigma(\xi) \otimes 1)(s \otimes v).$$

Since  $D$  is homogeneous, this determines the symbol at all points  $g \in G$  (see [Wal73] lemma 5.5.1).  $\sigma_D(\xi)$  is clearly an isomorphism for  $\xi \neq 0$ , since in this case we have that

$$\sigma(\xi)\sigma(-q(\xi)^{-1}\xi) = \sigma(-q(\xi)^{-1}\xi^2) = \sigma(1) = \text{Id}.$$

Hence  $D$  is elliptic. Therefore  $\ker D$  is finite-dimensional (see [LM89] theorem III.5.2). Since  $D$  is homogeneous,  $\ker D$  is invariant under the left regular action  $\tilde{l}$  of  $G$  on  $\Gamma(\underline{S \otimes V})$  and so the restriction of  $\tilde{l}$  to  $\ker D$  gives a finite-dimensional representation of  $G$  which we denote by  $\tilde{\pi}$ . Similarly, if  $n = \dim \mathfrak{p}$  is even, the restriction of the left regular action of  $G$  on  $\Gamma(\underline{S^\pm \otimes V})$  to  $\ker D^\pm$  give finite-dimensional representations  $\tilde{\pi}^\pm$  of  $G$ .

It is shown in [Par72] lemma 4.1 that we can fix a hermitian inner product  $\langle \cdot, \cdot \rangle_S$  on  $S$  with respect to which Clifford multiplication is skew-hermitian and  $\chi$  is unitary. Now by fixing a hermitian inner product  $\langle \cdot, \cdot \rangle_V$  on  $V$  such that  $\tau$  is unitary, we get an inner product on  $S \otimes V$  given by

$$\left\langle \sum_i s_i \otimes v_i, \sum_j s'_j \otimes v'_j \right\rangle = \sum_{i,j} \langle s_i, s'_j \rangle_S \langle v_i, v'_j \rangle_V$$

when  $s_i, s'_j \in S, v_i, v'_j \in V$  and with respect to this inner product,  $\chi \otimes \tau$  is unitary. We therefore have a hermitian inner product on each fibre of  $\underline{S \otimes V}$  and we get an inner product on  $\Gamma(\underline{S \otimes V})$  given by

$$\langle \varphi, \varphi' \rangle = \int_{G/H} \langle \varphi(x), \varphi'(x) \rangle dx$$

where we integrate with respect to a  $G$ -invariant measure on  $G/H$  (such a measure exists according to corollary A4 of Dupont [Dup03]). With respect to this inner product,  $\tilde{l}$  is unitary and lemma 4.2 of [Par72] shows that  $D$  is formally self-adjoint. If  $\dim \mathfrak{p}$  is even,  $D^+$  and  $D^-$  are formal adjoints of each other.

## 5 Symmetric spaces

### 5.1 Basic notions

In the rest of this paper we consider the special case of a compact symmetric space. In section 6.2 we see that in this case, the square of the Dirac operator has a particularly simple expression. This enables us to determine the representations in the kernel of the Dirac operator in section 6.3. We start by establishing some basic concepts concerning compact symmetric spaces. We then study the spinor representation in this case.

Let  $G$  be a compact connected semisimple Lie group with Lie algebra  $\mathfrak{g}$ . A Lie algebra automorphism  $\theta$  of  $\mathfrak{g}$  is called an involution of  $\mathfrak{g}$  if  $\theta^2 = \text{Id}$ . Let  $H$  be the analytic subgroup of  $G$  corresponding to the  $+1$  eigenspace  $\mathfrak{h}$  of  $\theta$ . Then  $G/H$  is called a symmetric space. We assume that  $\theta \neq \text{Id}$ , i.e.,  $\mathfrak{h}$  is a proper Lie subalgebra of  $\mathfrak{g}$ . Let  $\mathfrak{p}$  denote the  $-1$  eigenspace of  $\theta$  and let  $B$  be the Killing form on  $\mathfrak{g}$ .  $-B$  is a positive definite bilinear form on  $\mathfrak{g}$  and therefore it is an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . (See Knapp [Kna02] theorem 1.45 and corollary 4.26).  $\mathfrak{p}$  is the orthogonal complement of  $\mathfrak{h}$  with respect to  $\langle \cdot, \cdot \rangle$  and we have that

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{p}, \\ [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}. \end{aligned} \tag{5.1}$$

For  $h \in H$  and  $\xi, \xi' \in \mathfrak{g}$  we have that

$$\langle \text{Ad}(h)\xi, \text{Ad}(h)\xi' \rangle = -B(\text{Ad}(h)\xi, \text{Ad}(h)\xi') = -B(\xi, \xi') = \langle \xi, \xi' \rangle$$

(see [Kna02] proposition 1.119). Hence  $\text{Ad}(h)$  is orthogonal for all  $h$ . Since  $\mathfrak{h}$  is  $\text{Ad } H$ -invariant, we have that  $\mathfrak{p}$  is  $\text{Ad } H$ -invariant.

$H$  is not necessarily semisimple but since it is compact we have that  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus Z_{\mathfrak{h}}$  where  $Z_{\mathfrak{h}}$  is the center of  $\mathfrak{h}$  (see corollary 4.25 of [Kna02]). We note that

$$Z_{\mathfrak{h}} = [\mathfrak{h}, \mathfrak{h}]^{\perp} \cap \mathfrak{h}. \tag{5.2}$$

To see this, let  $\xi \in [\mathfrak{h}, \mathfrak{h}]^\perp \cap \mathfrak{h}$  and  $\xi' \in \mathfrak{h}$ . Then

$$\langle [\xi, \xi'], [\xi, \xi'] \rangle = \langle -\text{ad}(\xi')(\xi), [\xi, \xi'] \rangle = \langle \xi, \text{ad}(\xi')[\xi, \xi'] \rangle = \langle \xi, [\xi', [\xi, \xi']] \rangle = 0,$$

i.e.,  $[\xi, \xi'] = 0$  and therefore  $\xi \in Z_{\mathfrak{h}}$ . Hence  $[\mathfrak{h}, \mathfrak{h}]^\perp \cap \mathfrak{h} \subset Z_{\mathfrak{h}}$  and since  $\dim Z_{\mathfrak{h}} = \dim([\mathfrak{h}, \mathfrak{h}]^\perp \cap \mathfrak{h})$ , we conclude (5.2).

In the rest of this paper we make the assumption that the rank of  $G$  equals the rank of  $H$ . We now see that this implies that  $\dim \mathfrak{p}$  is even.  $\mathfrak{g}_{\mathbb{C}}$  has Cartan subalgebra  $\mathfrak{t}_{\mathbb{C}}$  where  $\mathfrak{t}$  is the Lie algebra of a maximal torus  $T$  in  $G$ . Let  $\mathfrak{t}'$  be the Lie algebra of a maximal torus  $T'$  of  $H$  and suppose that  $\dim \mathfrak{t} = \dim \mathfrak{t}'$ . Then we may assume that  $T = T'$ . Since  $\mathfrak{g}_{\mathbb{C}}$  is semisimple, we get a root space decomposition of  $\mathfrak{g}_{\mathbb{C}}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\pm\alpha}$$

where  $\Delta^+$  denotes the positive roots of  $\mathfrak{g}_{\mathbb{C}}$  and for each  $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ ,

$$\mathfrak{g}_{\alpha} = \{\xi \in \mathfrak{g}_{\mathbb{C}} \mid \text{ad}(X)\xi = \alpha(X)\xi \text{ for all } X \in \mathfrak{t}_{\mathbb{C}}\}.$$

Since for each root  $\alpha$ ,  $\mathfrak{g}_{\alpha}$  is the simultaneous eigenspace of  $\text{ad}(\mathfrak{t}_{\mathbb{C}})$  and  $\mathfrak{h}$  and  $\mathfrak{p}$  are  $\text{ad}(\mathfrak{t}_{\mathbb{C}})$ -invariant, we have that either  $\mathfrak{g}_{\alpha} \subset \mathfrak{h}_{\mathbb{C}}$  or  $\mathfrak{g}_{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$ . When  $\mathfrak{g}_{\alpha} \subset \mathfrak{h}_{\mathbb{C}}$ , it is the root space  $\mathfrak{h}_{\alpha}$  of  $\alpha$  in  $\mathfrak{h}_{\mathbb{C}}$  and we get that

$$\mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{h}}^+} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{p}_{\mathbb{C}} = \bigoplus_{\alpha \in \Delta_{\mathfrak{p}}^+} \mathfrak{g}_{\pm\alpha}$$

where the disjoint union  $\Delta_{\mathfrak{h}}^+ \cup \Delta_{\mathfrak{p}}^+ = \Delta^+$ . Since each root space is one-dimensional, we get that

$$\begin{aligned} \dim(\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}) &= \dim \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\pm\alpha} \text{ is even} \\ \dim(\mathfrak{h}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}) &= \dim \bigoplus_{\alpha \in \Delta_{\mathfrak{h}}^+} \mathfrak{g}_{\pm\alpha} \text{ is even.} \end{aligned}$$

Hence

$$\dim \mathfrak{p}_{\mathbb{C}} = \dim(\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}) = \dim(\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}) - \dim(\mathfrak{h}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}) \text{ is even.}$$

The following are examples of symmetric spaces with the above properties. We return to these examples in section 7.

*Example 5.1.* Let  $G = \text{SO}(3)$  and  $H = \text{SO}(2) \cong S^1$ .  $G$  and  $H$  are compact and connected and  $G$  is semisimple. (See [Kna02] section II.1 and [Kna02] proposition 1.136). The Lie algebras of  $G$  and  $H$  are given by

$$\begin{aligned} \mathfrak{g} &= \mathfrak{so}(3) = \{X \in M_3(\mathbb{R}) \mid X + X^t = 0\} \\ \mathfrak{h} &= \mathfrak{so}(2) = \{X \in M_2(\mathbb{R}) \mid X + X^t = 0\} \end{aligned}$$

and their complexifications are

$$\begin{aligned}\mathfrak{g}_{\mathbb{C}} &= \mathfrak{so}(3, \mathbb{C}) = \{X \in M_3(\mathbb{C}) \mid X + X^t = 0\} \\ \mathfrak{h}_{\mathbb{C}} &= \mathfrak{so}(2, \mathbb{C}) = \{X \in M_2(\mathbb{C}) \mid X + X^t = 0\}.\end{aligned}$$

Elements of  $H$  are of the form

$$h = \begin{pmatrix} \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & \cos \theta_1 \end{pmatrix}$$

where  $\theta_1 \in \mathbb{R}$  and we may think of  $H$  as a subgroup of  $G$  by identifying an element  $h \in H$  with the element

$$h = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(3).$$

Under this identification, we identify an element

$$X = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \in \mathfrak{h}$$

with the element

$$X = \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

Let  $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$  be defined by

$$\theta \begin{pmatrix} 0 & y_{12} & y_{13} \\ -y_{12} & 0 & y_{23} \\ -y_{13} & -y_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & y_{12} & -y_{13} \\ -y_{12} & 0 & -y_{23} \\ y_{13} & y_{23} & 0 \end{pmatrix}.$$

$\theta$  clearly is an involution of  $\mathfrak{g}$  with  $+1$  eigenspace  $\mathfrak{h}$ . Thus  $G/H = S^2$  is a symmetric space. The maximal torus of  $G$  and  $H$  is  $H$ , in particular,  $G$  and  $H$  have equal rank. Now define the element  $e_1 \in \mathfrak{h}_{\mathbb{C}}^*$  by

$$e_1 \begin{pmatrix} 0 & ix & 0 \\ -ix & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = x.$$

Then  $\Delta^+ = \Delta_{\mathfrak{p}}^+ = \{e_1\}$ .

*Example 5.2.* Let  $n = 2m \geq 4$  and  $G = \mathrm{SO}(n+1)$ ,  $H = \mathrm{SO}(n)$ . Note that  $G$  and  $H$  are compact, connected and semisimple for all  $n$ . (See [Kna02] section II.1 and proposition 1.136). The Lie algebras of  $G$  and  $H$  are given by

$$\begin{aligned}\mathfrak{g} &= \mathfrak{so}(n+1) = \{X \in M_{n+1}(\mathbb{R}) \mid X + X^t = 0\} \\ \mathfrak{h} &= \mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X + X^t = 0\}\end{aligned}$$

and their complexifications are

$$\begin{aligned}\mathfrak{g}_{\mathbb{C}} &= \mathfrak{so}(n+1, \mathbb{C}) = \{X \in M_{n+1}(\mathbb{C}) \mid X + X^t = 0\} \\ \mathfrak{h}_{\mathbb{C}} &= \mathfrak{so}(n, \mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid X + X^t = 0\}.\end{aligned}$$

Note that

$$\dim \mathfrak{g} = \frac{(n+1)^2 - (n+1)}{2} = \frac{n(n+1)}{2}, \quad \dim \mathfrak{h} = \frac{n^2 - n}{2} = \frac{n(n-1)}{2}.$$

We may think of  $H$  as a subgroup of  $G$  when we identify the element  $h \in \mathrm{SO}(n)$  with the matrix

$$h = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(n+1),$$

and under this identification, we identify an element  $X \in \mathfrak{h}$  with

$$X = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}.$$

We now consider the homogeneous space  $G/H$  which can be identified with the  $n$ -sphere  $S^n$  (see [War83] 3.65 (a)). In the following we show that  $G/H$  is a symmetric space. Any element of  $\mathfrak{g}$  is of the form

$$Y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,n} & y_{1,n+1} \\ \vdots & & \vdots & \vdots \\ y_{n,1} & \cdots & y_{n,n} & y_{n,n+1} \\ y_{n+1,1} & \cdots & y_{n+1,n} & 0 \end{pmatrix}$$

where  $(y_{i,j})_{1 \leq i,j \leq n} \in \mathfrak{h}$  and  $y_{i,n+1} = -y_{n+1,i}$  for  $i = 1, \dots, n$ . Now let  $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$  be given by

$$\theta(Y) = \begin{pmatrix} y_{1,1} & \cdots & y_{1,n} & -y_{1,n+1} \\ \vdots & & \vdots & \vdots \\ y_{n,1} & \cdots & y_{n,n} & -y_{n,n+1} \\ -y_{n+1,1} & \cdots & -y_{n+1,n} & 0 \end{pmatrix}.$$

It is easy to see that  $\theta$  is an involution of  $\mathfrak{g}$  where  $\mathfrak{h}$  is the  $+1$  eigenspace. Hence  $G/H$  is a symmetric space. We see that the elements of  $\mathfrak{p}$  are of the form

$$Y = \begin{pmatrix} 0 & \cdots & 0 & y_{1,n+1} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & y_{n,n+1} \\ -y_{1,n+1} & \cdots & -y_{n,n+1} & 0 \end{pmatrix}.$$

The dimension of  $\mathfrak{p}$  is

$$\dim \mathfrak{p} = \dim \mathfrak{g} - \dim \mathfrak{h} = n = 2m.$$

Because of the assumption that  $n = 2m$  is even,  $G$  and  $H$  are of equal rank since in this case  $\mathfrak{g}_{\mathbb{C}}$  and  $\mathfrak{h}_{\mathbb{C}}$  have the same Cartan subalgebra  $\mathfrak{t}_{\mathbb{C}}$  given by the

elements of  $\mathfrak{g}_{\mathbb{C}}$  of the form

$$X = \begin{pmatrix} 0 & ih_1 & & & \\ -ih_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & ih_m \\ & & & -ih_m & 0 \\ & & & & & 0 \end{pmatrix} \quad (5.3)$$

where  $h_1, \dots, h_m \in \mathbb{C}$ . Let  $\mathfrak{t}$  consist of the elements of the form (5.3) for which  $h_1, \dots, h_m \in i\mathbb{R}$ . Then  $\mathfrak{t}$  is the Lie algebra of the maximal torus  $T$  of  $G$  and  $H$ .  $T$  consists of the elements of  $\mathrm{SO}(n+1)$  of the form

$$t = \begin{pmatrix} \cos \theta_1 & \sin \theta_1 & & & \\ -\sin \theta_1 & \cos \theta_1 & & & \\ & & \ddots & & \\ & & & \cos \theta_m & \sin \theta_m \\ & & & -\sin \theta_m & \cos \theta_m \\ & & & & & 1 \end{pmatrix}$$

where  $\theta_1, \dots, \theta_m \in \mathbb{R}$ . For each  $X \in \mathfrak{t}_{\mathbb{C}}$  of the form given in (5.3), define

$$e_j(X) = h_j \quad \text{for } j \in \{1, \dots, m\}.$$

Then  $e_j \in \mathfrak{t}_{\mathbb{C}}^*$  and the positive roots of  $G$  and  $H$  are given by

$$\begin{aligned} \Delta^+ &= \{e_i \pm e_j \mid i < j\} \cup \{e_k\} \\ \Delta_{\mathfrak{h}}^+ &= \{e_i \pm e_j \mid i < j\}. \end{aligned}$$

## 5.2 The spinor representation

Assume that  $\mathrm{Ad}: H \rightarrow \mathrm{SO}(\mathfrak{p})$  lifts to  $\mathrm{Spin}(\mathfrak{p})$ . We then have the spinor representation  $\chi = \sigma \circ \bar{\rho}: H \rightarrow \mathrm{GL}(S)$  of  $H$  (as in (3.3)). The differential  $\chi_*$  of  $\chi$  at the identity is given by the composition

$$\mathfrak{h} \xrightarrow{\mathrm{ad}} \mathfrak{so}(\mathfrak{p}) \xrightarrow[\cong]{(\psi_*)^{-1}} \mathfrak{spin}(\mathfrak{p}) \xrightarrow{\sigma} \mathrm{End}(S). \quad (5.4)$$

Using proposition 2.10 we get the following result.

**Proposition 5.3.** *The weights of the spinor representation  $\chi: H \rightarrow \mathrm{GL}(S)$  for  $\dim \mathfrak{p} = 2m$  are given by*

$$\tilde{\lambda}_{\varepsilon} = \frac{1}{2} \sum_{k=1}^m \varepsilon_k \alpha_k, \quad \text{for } \varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{\pm 1\}^m$$

where  $\{\alpha_k \mid 1 \leq k \leq m\}$  is an enumeration of the roots of  $\Delta_{\mathfrak{p}}^+$ .



The weights of the representation  $\chi^+$  are  $\{\tilde{\lambda}_\varepsilon \mid \varepsilon \in E^+\}$  and the weights of  $\chi^-$  are  $\{\tilde{\lambda}_\varepsilon \mid \varepsilon \in E^-\}$  where  $E^\pm$  are as in (2.4).

The multiplicity of each  $\tilde{\lambda}_\varepsilon$  is the number of ways in which  $\tilde{\lambda}_\varepsilon$  can be written in the above form.

*Proof.* We think of  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  as the composition

$$H \xrightarrow{\text{Ad}} \text{SO}(\mathfrak{p}) \xrightarrow{\text{Id}} \text{SO}(\mathfrak{p})$$

where  $\text{Id}$  denotes the standard representation of  $\text{SO}(\mathfrak{p})$  on  $\mathfrak{p}$ . Since  $\text{Ad}(T)$  is a compact abelian Lie subgroup of  $\text{SO}(\mathfrak{p})$ , it must lie in a maximal torus  $T'$  of  $\text{SO}(\mathfrak{p})$ . So we get the following commutative diagram.

$$\begin{array}{ccccc} \mathfrak{t} & \xrightarrow{\text{ad}} & \mathfrak{t}' & \xrightarrow{\text{Id}_*} & \mathfrak{so}(\mathfrak{p}) \\ \exp \downarrow & & \downarrow \exp & & \downarrow \exp \\ T & \xrightarrow{\text{Ad}} & T' & \xrightarrow{\text{Id}} & \text{SO}(\mathfrak{p}) \end{array}$$

where  $\text{Id}_* = \text{ad}_{\text{Cl}} \circ \psi_*^{-1}$ . Let  $\bigoplus_{j=1}^m \mathfrak{p}_{\pm j}$  be the weight space decomposition of  $\mathfrak{p}_{\mathbb{C}}$  with respect to  $\text{Id}$ , i.e.,  $\mathfrak{p}_{\pm j}$  are all one-dimensional and

$$\text{Id}_*(X)v = \pm \eta_j(X)v \quad \text{for } v \in \mathfrak{p}_{\pm j}, X \in \mathfrak{t}'_{\mathbb{C}}.$$

We then have that

$$\text{Id}_* \circ \text{ad}(\xi)v = \pm \eta_j(\text{ad}(\xi))v \quad \text{for } v \in \mathfrak{p}_{\pm j}, \xi \in \mathfrak{t}_{\mathbb{C}}.$$

If  $\alpha_1, \dots, \alpha_m$  is an enumeration of the roots of  $\Delta_{\mathfrak{p}}^+$ , then  $\mathfrak{p}_{\mathbb{C}} = \bigoplus_{j=1}^m \mathfrak{g}_{\pm \alpha_j}$  where  $\mathfrak{g}_{\pm \alpha_j}$  are all one-dimensional and

$$\text{Id}_* \circ \text{ad}(\xi)v = \pm \alpha_j(\xi)v \quad \text{for } v \in \mathfrak{g}_{\pm \alpha_j}, \xi \in \mathfrak{t}_{\mathbb{C}}.$$

Hence by re-enumeration, we may assume that

$$(\eta_j \circ \text{ad})(\xi) = \alpha_j(\xi) \quad \text{for } \xi \in \mathfrak{t}_{\mathbb{C}}, j \in \{1, \dots, m\}.$$

Since  $\chi_* = \sigma \circ \psi_*^{-1} \circ \text{ad}$ , proposition 2.10 shows that

$$\begin{aligned} \chi_*(\xi)s &= \sigma(\psi_*^{-1} \circ \text{ad}(\xi))s = \lambda_\varepsilon(\psi_*^{-1} \circ \text{ad}(\xi))s \\ &= \frac{1}{2} \sum_{j=1}^m \varepsilon_j \eta_j(\text{ad}(\xi))s = \frac{1}{2} \sum_{j=1}^m \varepsilon_j \alpha_j(\xi)s \end{aligned}$$

for  $s \in S_\varepsilon, \xi \in \mathfrak{t}_{\mathbb{C}}$ . This proves the proposition.  $\square$

Now we compute an explicit expression for the spinor representation  $\chi_*$  which becomes useful in section 6.2 where we find the square of the Dirac operator. Let  $\{X_1, \dots, X_{2m}\}$  be an orthonormal basis of  $\mathfrak{p}$ . Recall that  $\{X_i X_j \in \text{Cl}(\mathfrak{p}) \mid i < j\}$  is a basis of  $\mathfrak{spin}(\mathfrak{p})$ . We have the following result (corresponding to lemma 2.1 of [Par72]).

**Lemma 5.4.** For all  $Y \in \mathfrak{h}$

$$\chi_*(Y) = \frac{1}{4} \sum_{k,l=1}^{2m} \langle [Y, X_k], X_l \rangle \sigma(X_k) \sigma(X_l).$$

*Proof.* We have the following commutative diagram.

$$\begin{array}{ccc} & & \mathfrak{so}(\mathfrak{p}) \\ & \nearrow \text{ad} & \uparrow \text{ad}_{C_1} \\ \mathfrak{h} & \xrightarrow{\bar{\rho}_*} & \mathfrak{spin}(\mathfrak{p}) \\ & \searrow \chi_* & \downarrow \sigma \\ & & \text{End}(S) \end{array} \quad (5.5)$$

Let  $Y \in \mathfrak{h}$ . We have that

$$\bar{\rho}_*(Y) = \sum_{k < l} C_{kl} X_k X_l$$

where  $C_{kl} \in \mathbb{R}$ . For each  $i \in \{1, \dots, 2m\}$  we get that

$$\begin{aligned} [Y, X_i] &= \text{ad}(Y)X_i = \text{ad}_{C_1}((\bar{\rho}_*)(Y))(X_i) \\ &= \sum_{k < l} C_{kl} (X_k X_l X_i - X_i X_k X_l) \\ &= \sum_{l > i} 2C_{il} X_l - \sum_{k < i} 2C_{ki} X_k. \end{aligned}$$

Here we have used the fact that

$$X_k X_l X_i - X_i X_k X_l = \begin{cases} 0 & \text{if } i \neq k, i \neq l \\ -2X_k & \text{if } i = l \\ 2X_l & \text{if } i = k \end{cases}.$$

Hence

$$\langle [Y, X_i], X_j \rangle = \begin{cases} 2C_{ij} & \text{for } i < j \\ -2C_{ij} & \text{for } i > j \\ 0 & \text{for } i = j \end{cases}.$$

This implies that

$$\bar{\rho}_*(Y) = \frac{1}{2} \sum_{k < l} \langle [Y, X_k], X_l \rangle X_k X_l$$

and that for  $k > l$  and  $i \in \{1, \dots, 2m\}$

$$\langle [Y, X_k], X_l \rangle X_k X_l = \langle [Y, X_l], X_k \rangle X_l X_k, \quad \langle [Y, X_i], X_i \rangle = 0.$$

Hence

$$\bar{\rho}_*(Y) = \frac{1}{4} \sum_{k,l=1}^{2m} \langle [Y, X_k], X_l \rangle X_k X_l$$

and the result follows from (5.5).  $\square$

### 5.3 The irreducible parts of $\chi$

We now study the irreducible parts of  $\chi$ . In section 6.1 we use this to find a simple expression for  $\chi_*(\Omega_H)$  where  $\Omega_H$  is the Casimir element of  $H$  and this is then used to find  $D^2$  in section 6.2.

Since  $\chi_*$  is a finite-dimensional representation of a complex reductive Lie algebra, it is completely reducible in sense that  $S$  splits into a direct sum of invariant subspaces and the restriction of  $\chi_*$  to each of these is irreducible (see [Kna02] theorem 5.29). Similarly,  $\chi_*^\pm$  are completely reducible. We now study this splitting in more detail. According to the theorem of the highest weight (theorem 5.110 in [Kna02]), each irreducible component of  $\chi$  is in one-to-one correspondence with the highest weight of that component. A weight of an irreducible component of  $\chi$  is of the form  $\tilde{\lambda}_\varepsilon$  as given in proposition 5.3. Now let  $W$  denote the Weyl group of  $G$  and let

$$W_1 = \{\sigma \in W \mid \Delta_{\mathfrak{h}}^+ \subset \sigma\Delta^+\}.$$

For each  $\sigma \in W_1$  we have that

$$\sigma\Delta^+ = \Delta_{\mathfrak{h}}^+ \cup \Delta_{\mathfrak{p}}^\sigma$$

where  $\Delta_{\mathfrak{p}}^\sigma = \{\varepsilon_1^\sigma \alpha_1, \dots, \varepsilon_m^\sigma \alpha_m\}$  for some  $(\varepsilon_1^\sigma, \dots, \varepsilon_m^\sigma) \in \{\pm 1\}^m$  and where  $\alpha_1, \dots, \alpha_m$  is an enumeration of the elements of  $\Delta_{\mathfrak{p}}^+$ .

**Lemma 5.5.** *Let  $W_H$  denote the Weyl group of  $H$ . The map*

$$\begin{array}{ccc} W_H \times W_1 & \rightarrow & W \\ (s, \sigma) & \mapsto & s\sigma \end{array}$$

*is a bijection.*

*Proof.* Let  $\Delta_{\mathfrak{h}} = \Delta_{\mathfrak{h}}^+ \cup (-\Delta_{\mathfrak{h}}^+)$  and  $\Delta_{\mathfrak{p}} = \Delta_{\mathfrak{p}}^+ \cup (-\Delta_{\mathfrak{p}}^+)$ . Suppose that  $w \in W$ . We have that  $w\Delta^+ = \Delta_{\mathfrak{h}}^w \cup \Delta_{\mathfrak{p}}^w$  where  $\Delta_{\mathfrak{h}}^w \subset \Delta_{\mathfrak{h}}$  and  $\Delta_{\mathfrak{p}}^w \subset \Delta_{\mathfrak{p}}$ . Since  $\Delta_{\mathfrak{h}}^w$  defines a system of positive roots of  $\mathfrak{h}_{\mathbb{C}}$ , there is a unique element  $s \in W_H$  such that  $s\Delta_{\mathfrak{h}}^+ = \Delta_{\mathfrak{h}}^w$  (see theorem 1.8 of Humphreys [Hum90]). Let  $\sigma = s^{-1}w$ . Then

$$\sigma\Delta^+ = s^{-1}(\Delta_{\mathfrak{h}}^w \cup \Delta_{\mathfrak{p}}^w) = \Delta_{\mathfrak{h}}^+ \cup s^{-1}\Delta_{\mathfrak{p}}^w.$$

So  $\sigma \in W_1$  and hence  $w = s\sigma$  for  $s \in W_H$  and  $\sigma \in W_1$ . This shows surjectivity.

Now suppose that  $s\sigma = s'\sigma'$  for  $s, s' \in W_H, \sigma, \sigma' \in W_1$ . We then have that  $\sigma'\sigma^{-1} = (s')^{-1}s \in W_H$ . Hence

$$\sigma'\sigma^{-1}\Delta_{\mathfrak{h}} = \Delta_{\mathfrak{h}}. \quad (5.6)$$

If  $\alpha \in \Delta_{\mathfrak{h}}^+$ , then  $\sigma^{-1}\alpha \in \Delta^+$  and therefore  $\sigma'\sigma^{-1}\alpha \in \Delta_{\mathfrak{h}}^+ \cup \Delta_{\mathfrak{p}}^{\sigma'}$ . But then (5.6) shows that  $\sigma'\sigma^{-1}\alpha \in \Delta_{\mathfrak{h}}^+$  and therefore

$$(s')^{-1}s\Delta_{\mathfrak{h}}^+ = \sigma'\sigma^{-1}\Delta_{\mathfrak{h}}^+ = \Delta_{\mathfrak{h}}^+$$

Hence  $(s')^{-1}s = 1$  (by theorem 1.8 of [Hum90]) which implies that  $s' = s, \sigma' = \sigma$ .  $\square$

Now let

$$\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

and let

$$\delta_{\mathfrak{h}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{h}}^+} \alpha, \quad \delta_{\mathfrak{p}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{p}}^+} \alpha.$$

For each  $\sigma \in W_1$ , let

$$\delta_{\mathfrak{p}}^{\sigma} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{p}}^{\sigma}} \alpha.$$

Let  $l$  denote the length function on the Weyl group  $W$ , i.e., if  $w \in W$ ,  $l(w)$  is the smallest number of reflections in simple roots the product of which is  $w$ . Corollary 1.7 of [Hum90] shows that  $l(w)$  is actually the number of positive roots sent to negative roots by  $w$ . The sign function  $\text{sgn}: W \rightarrow \{\pm 1\}$  on  $W$  is given by  $\text{sgn}(w) = \det w = (-1)^{l(w)}$ . Hence if  $\sigma \in W_1$ , then since  $l(\sigma)$  is the number of negative elements of  $\Delta_{\mathfrak{p}}^{\sigma}$ , we have that

$$\delta_{\mathfrak{p}}^{\sigma} \text{ is a weight of } \begin{cases} \chi^+ & \text{if } \text{sgn}(\sigma) = 1 \\ \chi^- & \text{if } \text{sgn}(\sigma) = -1 \end{cases}.$$

We have the following result.

**Lemma 5.6.** *For each  $\sigma \in W_1$ , the element  $\delta_{\mathfrak{p}}^{\sigma}$  is the highest weight of an irreducible component  $\tau_{\sigma}$  of  $\chi$ . When  $\sigma \neq \sigma'$ ,  $\tau_{\sigma}$  and  $\tau_{\sigma'}$  are not equivalent.*

*Proof.* Any weight  $\tilde{\lambda}_{\varepsilon}$  can be written as

$$\tilde{\lambda}_{\varepsilon} = \delta_{\mathfrak{p}}^{\sigma} - \sum_{\alpha \in \Phi_{\varepsilon}} \alpha$$

where  $\Phi_{\varepsilon} = \{\varepsilon_i^{\sigma} \alpha_i \in \Delta_{\mathfrak{p}}^{\sigma} \mid \varepsilon_i^{\sigma} \neq \varepsilon_i\}$ . Suppose that  $\delta_{\mathfrak{p}}^{\sigma}$  is not a highest weight. Then there would exist some  $\varepsilon \in \{\pm 1\}^m$  and an element  $\beta = \sum_i n_i \beta_i \neq 0$  where  $\{\beta_i\} \subset \Delta_{\mathfrak{h}}^+$  are simple roots of  $\mathfrak{h}_{\mathbb{C}}$  and  $n_i \in \{0, 2, \dots\}$  such that

$$\delta_{\mathfrak{p}}^{\sigma} + \beta = \delta_{\mathfrak{p}}^{\sigma} - \sum_{\alpha \in \Phi_{\varepsilon}} \alpha. \quad (5.7)$$

Now we may redefine the notion of positivity on  $\mathfrak{t}_{\mathbb{C}}^*$  such that  $\sigma \Delta^+$  are the positive roots of  $\mathfrak{g}_{\mathbb{C}}$ . With this notion of positivity we have that

$$\beta' = \sum_{\alpha \in \Phi_{\varepsilon}} \alpha \geq 0.$$

Since  $\Delta_{\mathfrak{h}}^+ \subset \sigma \Delta^+$ , we still have that  $\beta > 0$  and therefore  $\beta + \beta' > 0$ . However, (5.7) shows that  $\beta + \beta' = 0$ . This is a contradiction and therefore we conclude that  $\delta_{\mathfrak{p}}^{\sigma}$  is the highest weight of an irreducible component of  $\chi$ .

Now suppose that  $\delta_{\mathfrak{p}}^{\sigma} = \delta_{\mathfrak{p}}^{\sigma'}$ . Then  $\sigma\delta = \sigma'\delta$ , i.e.,

$$\langle \sigma^{-1}\sigma'\delta, \alpha \rangle = \langle \delta, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta^+$$

(see proposition 2.69 of [Kna02]). But then theorem 3.10.9 of Wallach [Wal73] shows that  $\sigma^{-1}\sigma' = 1$ , i.e.,  $\sigma = \sigma'$ . Hence  $\tau_{\sigma}$  and  $\tau_{\sigma'}$  are inequivalent for  $\sigma \neq \sigma'$ .  $\square$

We also have the following result.

**Lemma 5.7.** *The representations  $\chi^+$  and  $\chi^-$  have no weights in common.*

*Proof.* Let  $\{\beta_1, \dots, \beta_l\}$  be an enumeration of the simple roots of  $\Delta^+$  such that  $\{\beta_1, \dots, \beta_k\} \subset \Delta_{\mathfrak{h}}^+$  and  $\{\beta_{k+1}, \dots, \beta_l\} \subset \Delta_{\mathfrak{p}}^+$ . Note that  $\{\beta_{k+1}, \dots, \beta_l\} \neq \emptyset$  since we have assumed that  $\mathfrak{h} \neq \mathfrak{g}$ . Any positive element  $\alpha$  of  $\mathfrak{t}_{\mathbb{C}}^*$  can be written uniquely as

$$\alpha = \sum_{i=1}^l n_i \beta_i$$

where  $n_i \in \{0, 1, \dots\}$  and we call  $n(\alpha) = \sum_{i=1}^l n_i$  the level of  $\alpha$ . Let

$$n_{\mathfrak{p}}(\alpha) = \sum_{i=k+1}^l n_i \beta_i.$$

We claim that for  $\alpha \in \Delta_{\mathfrak{h}}^+$ ,  $n_{\mathfrak{p}}(\alpha)$  is even and for  $\alpha \in \Delta_{\mathfrak{p}}^+$ ,  $n_{\mathfrak{p}}(\alpha)$  is odd. We prove this by induction on the level  $n(\alpha)$  of  $\alpha$ . Suppose  $n(\alpha) = 1$ . If  $\alpha \in \Delta_{\mathfrak{h}}^+$ , then  $\alpha = \beta_i$  for some  $i \in \{1, \dots, k\}$  and therefore  $n_{\mathfrak{p}}(\alpha) = 0$ . If  $\alpha \in \Delta_{\mathfrak{p}}^+$ , then  $\alpha = \beta_i$  for some  $i \in \{k+1, \dots, l\}$  and hence  $n_{\mathfrak{p}}(\alpha) = 1$ . This proves the case  $n(\alpha) = 1$ . Now let  $\alpha \in \Delta^+$  with  $n(\alpha) > 1$  and suppose that the result holds for all elements of  $\Delta^+$  with level less than  $n(\alpha)$ . Since  $n(\alpha) = \sum_{i=1}^l n_i > 1$  and each  $n_i \in \{0, 1, \dots\}$ , there must be a  $j \in \{1, \dots, l\}$  such that

$$\alpha - \beta_j = (n_j - 1)\beta_j + \sum_{i \neq j} n_i \beta_i > 0.$$

Proposition 2.48 (e) of [Kna02] shows that  $\alpha - \beta_j \in \Delta^+$ . Hence by letting  $\beta = \alpha - \beta_j$  and  $\beta' = \beta_j$  we have written  $\alpha$  as a sum  $\alpha = \beta + \beta'$  where  $\beta, \beta' \in \Delta^+$  and  $n(\beta) < n(\alpha), n(\beta') < n(\alpha)$ . Since  $n_{\mathfrak{p}}(\alpha) = n_{\mathfrak{p}}(\beta) + n_{\mathfrak{p}}(\beta')$  the claim is proved if we can show the following:

$$\begin{aligned} \beta, \beta' \in \Delta_{\mathfrak{h}}^+ & \text{ implies } \alpha \in \Delta_{\mathfrak{h}}^+ \\ \beta, \beta' \in \Delta_{\mathfrak{p}}^+ & \text{ implies } \alpha \in \Delta_{\mathfrak{p}}^+ \\ \beta \in \Delta_{\mathfrak{h}}^+, \beta' \in \Delta_{\mathfrak{p}}^+ & \text{ implies } \alpha \in \Delta_{\mathfrak{p}}^+. \end{aligned} \tag{5.8}$$

Corollary 2.35 of [Kna02] shows that  $[\mathfrak{g}_\beta, \mathfrak{g}_{\beta'}] = \mathfrak{g}_\alpha$  and hence using the relations (5.1) we see that

$$\begin{aligned} \mathfrak{g}_\beta, \mathfrak{g}_{\beta'} \subset \mathfrak{h} & \text{ implies } \mathfrak{g}_\alpha \subset \mathfrak{h} \\ \mathfrak{g}_\beta, \mathfrak{g}_{\beta'} \subset \mathfrak{p} & \text{ implies } \mathfrak{g}_\alpha \subset \mathfrak{h} \\ \mathfrak{g}_\beta \subset \mathfrak{h}, \mathfrak{g}_{\beta'} \subset \mathfrak{p} & \text{ implies } \mathfrak{g}_\alpha \subset \mathfrak{p}. \end{aligned}$$

This shows (5.8) and we have therefore proved the claim.

Suppose that  $\chi^\pm$  have some weight in common. Then there are non-empty subsets  $\Phi, \Phi' \subset \Delta_{\mathfrak{p}}^+$  with  $|\Phi|$  even and  $|\Phi'|$  odd such that

$$\sum_{\alpha \in \Phi} \alpha = \sum_{\alpha \in \Phi'} \alpha.$$

But then we have that

$$\sum_{\alpha \in \Phi} n_{\mathfrak{p}}(\alpha) = \sum_{\alpha \in \Phi'} n_{\mathfrak{p}}(\alpha). \quad (5.9)$$

This is a contradiction since the above shows that the left hand side of (5.9) is even and the right hand side is odd.  $\square$

Using lemmas 5.6 and 5.7 we now find the splitting of  $\chi$  into irreducible parts. We have the following result (as in [Par72] lemma 2.2).

**Proposition 5.8.** *For each  $\sigma \in W_1$ , let  $\tau_\sigma$  denote the irreducible representation of  $H$  with highest weight  $\delta_{\mathfrak{p}}^\sigma$ . Then the multiplicity of each  $\tau_\sigma$  in  $\chi$  is one and*

$$\chi^+ = \bigoplus_{\sigma \in W_1^+} \tau_\sigma, \quad \chi^- = \bigoplus_{\sigma \in W_1^-} \tau_\sigma$$

where  $W_1^+ = \{\sigma \in W_1 \mid \text{sgn } \sigma = 1\}$ ,  $W_1^- = \{\sigma \in W_1 \mid \text{sgn } \sigma = -1\}$ .

*Proof.* We have that  $\tau^+ = \bigoplus_{\sigma \in W_1^+} \tau_\sigma$  is a subrepresentation of  $\chi^+$  and that  $\tau^- = \bigoplus_{\sigma \in W_1^-} \tau_\sigma$  is a subrepresentation of  $\chi^-$ . We show that

$$\text{Tr } \chi^+ - \text{Tr } \tau^+ = \text{Tr } \chi^- - \text{Tr } \tau^- = 0. \quad (5.10)$$

Recall that  $\delta_{\mathfrak{p}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{p}}^+} \alpha$ . We now have that

$$\begin{aligned} \text{Tr } \chi^+ - \text{Tr } \chi^- &= \left( \sum_{\Phi \subset \Delta_{\mathfrak{p}}^+, |\Phi| \text{ even}} e^{\delta_{\mathfrak{p}} - \sum_{\alpha \in \Phi} \alpha} \right) - \left( \sum_{\Phi \subset \Delta_{\mathfrak{p}}^+, |\Phi| \text{ odd}} e^{\delta_{\mathfrak{p}} - \sum_{\alpha \in \Phi} \alpha} \right) \\ &= e^{\delta_{\mathfrak{p}}} \sum_{\Phi \subset \Delta_{\mathfrak{p}}^+} (-1)^{|\Phi|} e^{\sum_{\alpha \in \Phi} \alpha} \\ &= \left( \prod_{\alpha \in \Delta_{\mathfrak{p}}^+} e^{\frac{\alpha}{2}} \right) \left( \prod_{\alpha \in \Delta_{\mathfrak{p}}^+} (1 - e^{-\alpha}) \right) \\ &= \prod_{\alpha \in \Delta_{\mathfrak{p}}^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}). \end{aligned}$$

Recall that  $\delta_{\mathfrak{h}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{h}}^+} \alpha$  and  $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ . Note that for  $\sigma \in W_1$  we have that  $\delta_{\mathfrak{p}}^{\sigma} + \delta_{\mathfrak{h}} = \sigma\delta$ . Using Weyl's character formula (theorem 5.75 of [Kna02]) and the bijection  $W_H \times W_1 \rightarrow W$  of lemma 5.5 we get that

$$\begin{aligned}
\mathrm{Tr} \tau^+ - \mathrm{Tr} \tau^- &= \left( \sum_{\sigma \in W_1^+} \sum_{s \in W_H} \mathrm{sgn}(s) e^{s(\delta_{\mathfrak{p}}^{\sigma} + \delta_{\mathfrak{h}})} \right) \left( \sum_{s \in W_H} \mathrm{sgn}(s) e^{s\delta_{\mathfrak{h}}} \right)^{-1} \\
&\quad - \left( \sum_{\sigma \in W_1^-} \sum_{s \in W_H} \mathrm{sgn}(s) e^{s(\delta_{\mathfrak{p}}^{\sigma} + \delta_{\mathfrak{h}})} \right) \left( \sum_{s \in W_H} \mathrm{sgn}(s) e^{s\delta_{\mathfrak{h}}} \right)^{-1} \\
&= \left( \sum_{\sigma \in W_1} \mathrm{sgn}(\sigma) \sum_{s \in W_H} \mathrm{sgn}(s) e^{s\sigma\delta} \right) \left( \sum_{s \in W_H} \mathrm{sgn}(s) e^{s\delta_{\mathfrak{h}}} \right)^{-1} \\
&= \left( \sum_{w \in W} \mathrm{sgn}(w) e^{w\delta} \right) \left( \sum_{s \in W_H} \mathrm{sgn}(s) e^{s\delta_{\mathfrak{h}}} \right)^{-1} \\
&= \prod_{\alpha \in \Delta^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \prod_{\alpha \in \Delta_{\mathfrak{h}}^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})^{-1} \\
&= \prod_{\alpha \in \Delta_{\mathfrak{p}}^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) = \mathrm{Tr} \chi^+ - \mathrm{Tr} \chi^-.
\end{aligned}$$

We therefore conclude that

$$\mathrm{Tr} \chi^+ - \mathrm{Tr} \tau^+ = \mathrm{Tr} \chi^- - \mathrm{Tr} \tau^-.$$

If this does not vanish,  $\chi^+$  and  $\chi^-$  must have some irreducible component in common and therefore some weight of  $\chi^+$  must also be a weight of  $\chi^-$ . This contradicts lemma 5.6 and we have therefore proved (5.10) and hence that  $\chi^+ = \tau^+$  and  $\chi^- = \tau^-$ . Since the  $\tau_{\sigma}$ 's are inequivalent, the multiplicity of each of them is one.  $\square$

## 6 The Dirac operator on symmetric spaces

### 6.1 The action of Casimir elements

In this section we study the action of the Casimir element of  $G$  on the sections of an induced bundle  $\underline{V}$  and the action of the Casimir element of  $H$  on the space of spinors  $S$ . Both of these actions are important in determining the expression for the square of the Dirac operator.

Let  $\{Y_1, \dots, Y_r, X_1, \dots, X_{2m}\}$  be an orthonormal basis of  $\mathfrak{g}$  such that  $\{Y_1, \dots, Y_r\}$  is a basis of  $\mathfrak{h}$  and  $\{X_1, \dots, X_{2m}\}$  is a basis of  $\mathfrak{p}$ . We then have the Casimir

element  $\Omega_H \in U(\mathfrak{h}_{\mathbb{C}})$  of  $H$  and the Casimir element  $\Omega \in U(\mathfrak{g}_{\mathbb{C}})$  of  $G$  given by

$$\Omega_H = \sum_{i=1}^r -Y_i^2, \quad \Omega = \sum_{i=1}^r -Y_i^2 + \sum_{i=1}^{2m} -X_i^2$$

where  $U(\mathfrak{h}_{\mathbb{C}})$  and  $U(\mathfrak{g}_{\mathbb{C}})$  denote the universal enveloping algebras of  $\mathfrak{h}_{\mathbb{C}}$  and  $\mathfrak{g}_{\mathbb{C}}$  respectively.

Recall that we have an action  $\tilde{l}_*$  of  $\mathfrak{g}$  on  $\Gamma(\underline{V})$  given by

$$\tilde{l}_*(\xi)(\tilde{\varphi} \otimes v)(g') = \frac{d}{dt}|_{t=0} \tilde{\varphi}(\exp(-t\xi)g') \otimes v = \tilde{\xi}\tilde{\varphi} \otimes v(g') \quad (6.1)$$

where  $\tilde{\xi}$  denotes the right invariant vector field on  $G$  induced by  $\xi$ . We may identify the tangent bundle of  $G$  with the induced vector bundle  $G \times \mathfrak{g}$  under the trivial representation  $\{e\}$  of  $G$  and a vector  $X = (l_g)_*\eta \in T_g(G)$  is identified with  $(g, \eta)$ . In order to describe the vector field  $\tilde{\xi}$  in these terms, we would like to find  $\eta(g) \in \mathfrak{g}$  for each  $g \in G$  such that

$$\tilde{\xi}\tilde{\varphi}(g) = (l_g)_*\eta(g) \quad .$$

We have that

$$\begin{aligned} \eta(g)\tilde{\varphi}(e) &= (l_{g^{-1}})_*\tilde{\xi}\tilde{\varphi}(g) = \frac{d}{dt}|_{t=0} \tilde{\varphi}(g^{-1}\exp(-t\xi)g) \\ &= \frac{d}{dt}|_{t=0} \tilde{\varphi}(\exp(-t\operatorname{Ad}(g^{-1})\xi)) \\ &= \operatorname{Ad}(g^{-1})\xi\tilde{\varphi}(e). \end{aligned}$$

Hence we identify  $\tilde{\xi}(g)$  with  $\operatorname{Ad}(g^{-1})\xi$ . Proposition 5.24 of [Kna02] shows that  $\Omega$  is in the center of  $U(\mathfrak{g}_{\mathbb{C}})$ . So if we denote the element  $\sum_{i=1}^r -Y_i^2 + \sum_{i=1}^{2m} -X_i^2$  by  $\tilde{\Omega}$ , then we may identify  $\tilde{\Omega}(g)$  with  $\operatorname{Ad}(g^{-1})\Omega = \Omega$ . Therefore

$$\begin{aligned} \tilde{\Omega}\tilde{\varphi}(g) &= \Omega\tilde{\varphi}(e) \\ &= \frac{d}{dt}|_{t=0} \sum_{i=1}^r -\tilde{\varphi}(\exp(-2tY_i)) + \sum_{i=1}^{2m} -\tilde{\varphi}(\exp(-2tX_i)) \\ &= \left( \sum_{i=1}^r -(\tilde{r}_*(Y_i))^2 + \sum_{i=1}^{2m} -(\tilde{r}_*(X_i))^2 \right) \tilde{\varphi}(e) \\ &= \tilde{r}_*(\Omega)\tilde{\varphi}(g) \end{aligned}$$

where  $\tilde{r}_*$  is the differential of the right regular action of  $\mathfrak{g}$  on  $C^\infty(G)$  which we extend to an action of  $U(\mathfrak{g}_{\mathbb{C}})$ . Combining this with (6.1) we get the following result.

**Lemma 6.1.** *Let  $\Omega$  be the Casimir element of  $\mathfrak{g}$ . For each section  $\tilde{\varphi} \otimes v$  of an induced bundle  $\underline{V}$  we have that*

$$\tilde{l}_*(\Omega)(\tilde{\varphi} \otimes v) = (\tilde{r}_*(\Omega) \otimes 1)(\tilde{\varphi} \otimes v)$$



where  $\tilde{l}_*$  denotes the left regular action of  $U(\mathfrak{g}_{\mathbb{C}})$  on  $\Gamma(\underline{V})$  and  $\tilde{r}_*$  denotes the right regular action of  $U(\mathfrak{g}_{\mathbb{C}})$  on  $C^\infty(G)$ .

Now we see how  $\chi_*(\Omega_H)$  acts on  $S$ . We have the following result.

**Lemma 6.2.** *Let  $\Omega_H$  be the Casimir element of  $\mathfrak{h}$ .  $\chi_*(\Omega_H)$  acts on  $S$  as scalar multiplication by  $\langle \delta, \delta \rangle - \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle$ .*

*Proof.* By proposition 5.8 we know that  $\chi_*$  is a sum of irreducible parts  $(\tau_\sigma)_*$  where  $\sigma \in W_1$ . Suppose that  $H$  is semisimple. By proposition 5.28 of [Kna02] we have that  $(\tau_\sigma)_*(\Omega_H)$  acts as scalar multiplication by  $\langle \delta_{\mathfrak{p}}^\sigma, \delta_{\mathfrak{p}}^\sigma + 2\delta_{\mathfrak{h}} \rangle$ . Since

$$\begin{aligned} \langle \delta_{\mathfrak{p}}^\sigma, \delta_{\mathfrak{p}}^\sigma + 2\delta_{\mathfrak{h}} \rangle &= \langle \delta_{\mathfrak{p}}^\sigma + \delta_{\mathfrak{h}}, \delta_{\mathfrak{p}}^\sigma + \delta_{\mathfrak{h}} \rangle - \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle = \langle \sigma\delta, \sigma\delta \rangle - \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle \\ &= \langle \delta, \delta \rangle - \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle \end{aligned}$$

for all  $\sigma \in W_1$ ,  $\chi_*(\Omega_H)$  acts as scalar multiplication by  $\langle \delta, \delta \rangle - \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle$  on  $S$ . Suppose  $H$  is not semisimple, i.e.,  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus Z_{\mathfrak{h}}$  where  $Z_{\mathfrak{h}} \neq 0$ . Since (5.2) shows that  $Z_{\mathfrak{h}} = [\mathfrak{h}, \mathfrak{h}]^\perp \cap \mathfrak{h}$ , we may assume that

$$[\mathfrak{h}, \mathfrak{h}] = \text{span}\{Y_1, \dots, Y_t\}, \quad Z_{\mathfrak{h}} = \text{span}\{Y_{t+1}, \dots, Y_r\}$$

for some  $t$ . Let

$$\Omega_{[\mathfrak{h}, \mathfrak{h}]} = \sum_{i=1}^t -Y_i^2, \quad \Omega_{Z_{\mathfrak{h}}} = \sum_{i=t+1}^r -Y_i^2.$$

The irreducible representations of  $\mathfrak{h}$  are the irreducible representations of  $[\mathfrak{h}, \mathfrak{h}]$  extended to  $\mathfrak{h}$  by being 0 on  $Z_{\mathfrak{h}}$  and the irreducible representations of  $Z_{\mathfrak{h}}$  extended to  $\mathfrak{h}$  by being 0 on  $[\mathfrak{h}, \mathfrak{h}]$ . If  $(\tau_\sigma)_*$  is an irreducible representation of  $[\mathfrak{h}, \mathfrak{h}]$  with highest weight  $\delta_{\mathfrak{p}}^\sigma$ , then as in the semisimple case we get that

$$(\tau_\sigma)_*(\Omega_H) = (\tau_\sigma)_*(\Omega_{[\mathfrak{h}, \mathfrak{h}]}) = \langle \delta, \delta \rangle - \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle.$$

The irreducible representations of  $Z_{\mathfrak{h}}$  are just linear functionals on  $Z_{\mathfrak{h}}$  times  $i$  and each representation has itself as a weight. So if  $(\tau_\sigma)_*$  is an irreducible representation of  $Z_{\mathfrak{h}}$  with highest weight  $\delta_{\mathfrak{p}}^\sigma$ , then

$$(\tau_\sigma)_*(\Omega_H) = (\tau_\sigma)_*(\Omega_{Z_{\mathfrak{h}}}) = |\delta_{\mathfrak{p}}^\sigma|^2.$$

Since  $(Z_{\mathfrak{h}})_{\mathbb{C}} \subset \mathfrak{t}_{\mathbb{C}}$ , we have that  $(Z_{\mathfrak{h}})_{\mathbb{C}} \perp \mathfrak{g}_\alpha$  for all  $\alpha \in \Delta_{\mathfrak{h}}^+$ . Hence  $\langle \delta_{\mathfrak{p}}^\sigma, \alpha \rangle = 0$  for all  $\alpha \in \Delta_{\mathfrak{h}}^+$  and therefore  $|\delta_{\mathfrak{p}}^\sigma|^2 = \langle \delta_{\mathfrak{p}}^\sigma, \delta_{\mathfrak{p}}^\sigma + 2\delta_{\mathfrak{h}} \rangle$ .  $\square$

## 6.2 The square of the Dirac operator

We now show that in the symmetric case, the square of the Dirac operator has a simple expression, namely, it consists of a constant term plus the left regular action of the Casimir element of  $G$ . Note that the restriction of  $D^2$  to  $\Gamma(\underline{S^+} \otimes V)$  is  $D^- \circ D^+$  and that the restriction of  $D^2$  to  $\Gamma(\underline{S^-} \otimes V)$  is  $D^+ \circ$

$D^-$ . The expression for  $D^2$  shows that elements in the kernels of  $D^\pm$  are eigenvectors of the left regular action of the Casimir element of  $G$ . We therefore obtain a criterion that must be met by the highest weights of the irreducible subrepresentations which appear in the kernels of  $D^\pm$  (see corollary 6.5). This is an important step in determining the representations  $\tilde{\pi}^\pm$  on the kernels of  $D^\pm$  in section 6.3.

An element of  $\mathfrak{t}_\mathbb{C}^*$  is said to be analytically integral if it induces a character on the maximal torus  $T$  of  $H$  and  $G$ . Let

$$\mathcal{F} = \{ \mu \in \mathfrak{t}_\mathbb{C}^* \mid \mu \text{ is analytically integral} \}.$$

According to the theorem of the highest weight (theorem 5.110 of [Kna02]), the irreducible representations of  $H$  stand in one-to-one correspondance with the dominant analytically integral forms of  $H$ , i.e., the elements

$$\mathcal{F}_H = \left\{ \mu \in \mathcal{F} \mid \langle \mu, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta_\mathfrak{h}^+ \right\},$$

the correspondance being that  $\mu \in \mathcal{F}_H$  is the highest weight of the representation. Similarly, the irreducible representations of  $G$  stand in one-to-one correspondance with

$$\mathcal{F}_G = \left\{ \mu \in \mathcal{F} \mid \langle \mu, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta^+ \right\}.$$

Since  $\delta_\mathfrak{p}$  is the highest weight of an irreducible component of  $\chi^+$ ,  $\delta_\mathfrak{p}$  is an element of  $\mathcal{F}_H$  so if  $\mu$  is any element of  $\mathcal{F}_H$ , then  $\lambda = \mu - \delta_\mathfrak{p} \in \mathcal{F}$ . We now have the following result (corresponding to proposition 3.2 of [Par72]).

**Proposition 6.3.** *Let  $G$  and  $H$  be compact connected Lie groups of equal rank where  $G$  is semisimple and  $G/H$  is a symmetric space and suppose that  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  lifts to  $\text{Spin}(\mathfrak{p})$ . For  $\mu \in \mathcal{F}_H$ , let  $(V_\mu, \tau_\mu)$  be the irreducible complex representation of  $H$  with highest weight  $\mu$ . Let  $\lambda = \mu - \delta_\mathfrak{p}$ . Then the square of the Dirac operator on  $\Gamma(\underline{S \otimes V_\mu})$  is given by*

$$D_\mu^2 = (\tilde{l}_\mu)_*(\Omega) - \langle \lambda + 2\delta, \lambda \rangle 1$$

where  $(\tilde{l}_\mu)_*(\Omega)$  denotes the left regular action of the Casimir element of  $\mathfrak{g}$  on  $\Gamma(\underline{S \otimes V_\mu})$ .

*Proof.* For notational convenience we omit the subscript  $\mu$  in this proof. Proposition 4.2 showed that in terms of the isomorphism (4.1),  $D$  is given by

$$D = \sum_{i=1}^{2m} \tilde{r}_*(X_i) \otimes \sigma(X_i) \otimes 1.$$

Since

$$\sigma(X_i)^2 = -1, \quad \sigma(X_i)\sigma(X_j) = -\sigma(X_j)\sigma(X_i) \text{ for } i \neq j$$

we get that

$$\begin{aligned}
D^2 &= \sum_{i=1}^{2m} \tilde{r}_*(X_i)^2 \otimes \sigma(X_i)^2 \otimes 1 + \sum_{i \neq j} \tilde{r}_*(X_i) \tilde{r}_*(X_j) \otimes \sigma(X_i) \sigma(X_j) \otimes 1 \\
&= \sum_{i=1}^{2m} \tilde{r}_*(X_i)^2 \otimes (-1) \otimes 1 + \frac{1}{2} \sum_{i,j=1}^{2m} \tilde{r}_*[X_i, X_j] \otimes \sigma(X_i) \sigma(X_j) \otimes 1. \quad (6.2)
\end{aligned}$$

Since  $[X_i, X_j] \in \mathfrak{h}$  for all  $i, j$  and  $\text{ad}$  is skew-symmetric on  $\mathfrak{g}$  with respect to  $\langle \cdot, \cdot \rangle$ , we get that

$$\begin{aligned}
[X_i, X_j] &= \sum_{q=1}^r \langle [X_i, X_j], Y_q \rangle Y_q = \sum_{q=1}^r \langle \text{ad}(X_i) X_j, Y_q \rangle Y_q \\
&= \sum_{q=1}^r \langle -\text{ad}(X_i) Y_q, X_j \rangle Y_q = \sum_{q=1}^r \langle [Y_q, X_i], X_j \rangle Y_q.
\end{aligned}$$

Substituting this into (6.2) and using lemma 5.4 gives us that

$$\begin{aligned}
D^2 &= \sum_{i=1}^{2m} \tilde{r}_*(X_i)^2 \otimes (-1) \otimes 1 + \frac{1}{2} \sum_{i,j=1}^{2m} \sum_{q=1}^r \langle [Y_q, X_i], X_j \rangle \tilde{r}_*(Y_q) \otimes \sigma(X_i) \sigma(X_j) \otimes 1 \\
&= \sum_{i=1}^{2m} \tilde{r}_*(X_i)^2 \otimes (-1) \otimes 1 + 2 \sum_{q=1}^r \tilde{r}_*(Y_q) \otimes \chi_*(Y_q) \otimes 1.
\end{aligned}$$

Now observe that

$$\begin{aligned}
(\tilde{r}_* \otimes \chi_*)(Y_q)^2 \otimes 1 &= (\tilde{r}_*(Y_q) \otimes 1 \otimes 1 + 1 \otimes \chi_*(Y_q) \otimes 1)^2 \\
&= \tilde{r}_*(Y_q)^2 \otimes 1 \otimes 1 + 1 \otimes \chi_*(Y_q)^2 \otimes 1 + 2\tilde{r}_*(Y_q) \otimes \chi_*(Y_q) \otimes 1.
\end{aligned}$$

Differentiating the  $H$ -invariance condition on  $C^\infty(G) \otimes S \otimes V$  of (4.1), we get that

$$\tilde{r}_*(Y_q) \otimes 1 \otimes 1 + 1 \otimes \chi_*(Y_q) \otimes 1 + 1 \otimes 1 \otimes \tau_*(Y_q) = 0$$

and therefore

$$(\tilde{r}_* \otimes \chi_*)(Y_q) \otimes 1 = \tilde{r}_*(Y_q) \otimes 1 \otimes 1 + 1 \otimes \chi_*(Y_q) \otimes 1 = -1 \otimes 1 \otimes \tau_*(Y_q).$$

Hence

$$\begin{aligned}
2r_*(Y_q) \otimes \chi_*(Y_q) \otimes 1 &= -\tilde{r}_*(Y_q)^2 \otimes 1 \otimes 1 - 1 \otimes \chi_*(Y_q)^2 \otimes 1 + (\tilde{r}_* \otimes \chi_*)(Y_q)^2 \otimes 1 \\
&= -\tilde{r}_*(Y_q)^2 \otimes 1 \otimes 1 - 1 \otimes \chi_*(Y_q)^2 \otimes 1 + 1 \otimes 1 \otimes \tau_*(Y_q)^2
\end{aligned}$$

and therefore

$$\begin{aligned}
D^2 &= \sum_{i=1}^{2m} -\tilde{r}_*(X_i)^2 \otimes 1 \otimes 1 \\
&\quad + \sum_{q=1}^r (-\tilde{r}_*(Y_q)^2 \otimes 1 \otimes 1 - 1 \otimes \chi_*(Y_q)^2 \otimes 1 + 1 \otimes 1 \otimes \tau_*(Y_q)^2) \\
&= \tilde{r}_*(\Omega) \otimes 1 \otimes 1 - (1 \otimes 1 \otimes \tau_*(\Omega_H) - 1 \otimes \chi_*(\Omega_H) \otimes 1).
\end{aligned}$$

We know from lemma 6.2 that  $\chi_*(\Omega_H) = (\langle \delta, \delta \rangle - \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle)1$  and since  $\tau$  is an irreducible representation of  $H$  with highest weight  $\mu = \lambda + \delta_{\mathfrak{p}}$ ,  $\tau_*(\Omega_H) = \langle \lambda + \delta_{\mathfrak{p}} + 2\delta_{\mathfrak{h}}, \lambda + \delta_{\mathfrak{p}} \rangle 1$ . So

$$\begin{aligned} 1 \otimes 1 \otimes \tau_*(\Omega_H) - 1 \otimes \chi_*(\Omega_H) \otimes 1 &= (\langle \lambda + \delta_{\mathfrak{p}} + 2\delta_{\mathfrak{h}}, \lambda + \delta_{\mathfrak{p}} \rangle - \langle \delta, \delta \rangle + \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle) 1 \\ &= (\langle \lambda + 2\delta - \delta_{\mathfrak{p}}, \lambda + \delta_{\mathfrak{p}} \rangle - \langle \delta, \delta \rangle + \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle) 1. \end{aligned}$$

Now observe that

$$\begin{aligned} \langle \lambda + 2\delta - \delta_{\mathfrak{p}}, \lambda + \delta_{\mathfrak{p}} \rangle - \langle \delta, \delta \rangle + \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle &= \langle \lambda + 2\delta, \lambda \rangle + \langle 2\delta - \delta_{\mathfrak{p}}, \delta_{\mathfrak{p}} \rangle - \langle \delta, \delta \rangle + \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle \\ &= \langle \lambda + 2\delta, \lambda \rangle + 2(\langle \delta_{\mathfrak{p}}, \delta_{\mathfrak{p}} \rangle + \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{p}} \rangle) - \langle \delta_{\mathfrak{p}}, \delta_{\mathfrak{p}} \rangle \\ &\quad - (\langle \delta_{\mathfrak{p}}, \delta_{\mathfrak{p}} \rangle + \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle + 2\langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{p}} \rangle) + \langle \delta_{\mathfrak{h}}, \delta_{\mathfrak{h}} \rangle \\ &= \langle \lambda + 2\delta, \lambda \rangle. \end{aligned}$$

We conclude that the square of the Dirac operator on  $\Gamma(\underline{S \otimes V_{\mu}})$  in terms of (4.1) is given by

$$D^2 = \tilde{r}_*(\Omega) \otimes 1 \otimes 1 - \langle \lambda + 2\delta, \lambda \rangle 1.$$

Now using lemma 6.1 we get the the desired result.  $\square$

We now show that the assumption that  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  lifts to  $\text{Spin}(\mathfrak{p})$  is not always necessary. Assume that  $G$  and  $H$  are compact connected Lie groups of equal rank such that  $G$  is semisimple and  $G/H$  is a symmetric space. Let  $\psi_H: \tilde{H} \rightarrow H$  denote the universal covering homomorphism of  $H$ . The kernel  $\ker \psi_H \cong \pi_1(H)$  is a discrete subgroup of the center  $Z(\tilde{H})$  of  $\tilde{H}$ . If  $\psi_G: \tilde{G} \rightarrow G$  denotes the universal covering homomorphism of  $G$  and  $\iota: H \rightarrow G$  is the inclusion of  $H$  in  $G$ , then  $\iota$  lifts to  $\tilde{\iota}: \tilde{H} \rightarrow \tilde{G}$ , i.e., the following diagram commutes

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\tilde{\iota}} & \tilde{G} \\ \downarrow \psi_H & & \downarrow \psi_G \\ H & \xrightarrow{\iota} & G \end{array}.$$

Suppose  $\dim \mathfrak{p} \geq 3$ . Then  $\psi: \text{Spin}(\mathfrak{p}) \rightarrow \text{SO}(\mathfrak{p})$  is the universal covering homomorphism of  $\text{SO}(\mathfrak{p})$  and therefore  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  lifts to a homomorphism  $\widetilde{\text{Ad}}$  as follows

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\widetilde{\text{Ad}}} & \text{Spin}(\mathfrak{p}) \\ \downarrow \psi_H & & \downarrow \psi \\ H & \xrightarrow{\text{Ad}} & \text{SO}(\mathfrak{p}) \end{array}$$

Consider  $\widetilde{\text{Ad}}|_{\ker \psi_H}: \ker \psi_H \rightarrow \{\pm 1\}$  and let  $K = \ker \widetilde{\text{Ad}}|_{\ker \psi_H}$ . Then  $K$  is a subgroup of  $\ker \psi_H$  of index either 1 or 2 contained in  $Z(\tilde{H})$ . So if  $H_1 = \tilde{H}/K$ , then  $H_1$  is either a two-fold cover of  $H$  or it is  $H$  itself. We get the following

diagram

$$\begin{array}{ccc}
\tilde{H} & \xrightarrow{\tilde{\iota}} & \tilde{G} \\
\downarrow \psi_H & & \downarrow \psi_G \\
H_1 & \xrightarrow{\iota_1} & G_1 \\
\downarrow & & \downarrow \\
H & \xrightarrow{\iota} & G
\end{array}$$

where  $G_1 = \tilde{G}/\tilde{\iota}(K)$  and  $\iota_1$  lifts  $\iota$ . When  $\iota_1$  is injective, then  $H_1$  lies naturally inside  $G_1$  and  $G_1/H_1 = G/H$ . Lifting  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  to the map  $\text{Ad}_1: H_1 \rightarrow \text{Spin}(\mathfrak{p})$  we get the spinor representations  $\chi = \sigma \circ \text{Ad}_1: H_1 \rightarrow \text{End}(S)$  and  $\chi^\pm = \sigma^\pm \circ \text{Ad}_1: H_1 \rightarrow \text{End}(S^\pm)$  of  $H_1$ .

Similarly, if  $\dim \mathfrak{p} = 2$  then  $\mathbb{R}$  is the universal cover of  $\text{SO}(\mathfrak{p})$  and if  $\psi \circ \phi: \mathbb{R} \rightarrow \text{SO}(\mathfrak{p})$  denotes the covering homomorphism,  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  lifts as follows

$$\begin{array}{ccc}
\tilde{H} & \xrightarrow{\tilde{\text{Ad}}} & \mathbb{R} \\
\downarrow \psi_H & & \downarrow \phi \\
H & \xrightarrow{\text{Ad}} & \text{Spin}(\mathfrak{p}) \\
& & \downarrow \psi \\
& & \text{SO}(\mathfrak{p})
\end{array}$$

We have that  $\ker \phi = 2\mathbb{Z}$ . Consider  $\text{Ad}|_{\ker \psi_H}: \ker \psi_H \rightarrow \mathbb{Z}$  and let  $K = \ker \text{Ad}|_{\ker \psi_H}$ . Then as before  $H_1 = \tilde{H}/K$  is either the trivial or two-fold cover of  $H$  and if  $G_1 = \tilde{G}/K$  and  $\iota_1$ , then  $G_1/H_1 = G/H$  and the lifting  $\text{Ad}_1: H_1 \rightarrow \text{Spin}(\mathfrak{p})$  of  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  gives the spinor representations  $\chi = \sigma \circ \text{Ad}_1: H_1 \rightarrow \text{End}(S)$  and  $\chi^\pm = \sigma^\pm \circ \text{Ad}_1: H_1 \rightarrow \text{End}(S^\pm)$  of  $H_1$ .

We conclude that by going to the covering group  $H_1$  of  $H$  which is either a two-fold cover or  $H$  itself, we in some cases (namely when  $\iota_1$  is injective) get a spin structure on  $G/H = G_1/H_1$ , even if  $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{p})$  does not lift to  $\text{Spin}(\mathfrak{p})$ . In the rest of this paper, whenever  $G_1$  and  $H_1$  are mentioned, we assume that  $\iota_1$  is injective.

Since  $G$  and  $H$  are compact,  $G_1$  and  $H_1$  are compact. Note that the Lie algebras of  $G_1$  and  $H_1$  are  $\mathfrak{g}$  and  $\mathfrak{h}$  respectively and that the complexification of the Lie algebra of the maximal torus  $T_1$  of  $H_1$  and  $G_1$  is  $\mathfrak{t}_{\mathbb{C}}$ . As before, we define the following

$$\begin{aligned}
\mathcal{F}_1 &= \{\mu \in \mathfrak{t}_{\mathbb{C}}^* \mid \mu \text{ induces a character on } T_1\} \\
\mathcal{F}_{H_1} &= \{\mu \in \mathcal{F}_1 \mid \langle \mu, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta_{\mathfrak{h}}^+\} \\
\mathcal{F}_{G_1} &= \{\mu \in \mathcal{F}_1 \mid \langle \mu, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta^+\}.
\end{aligned}$$

Now let

$$\mathcal{F}'_{H_1} = \{\mu \in \mathcal{F}_{H_1} \mid \mu - \delta_{\mathfrak{p}} \in \mathcal{F}\}.$$

Note that if  $\mu$  induces a character on  $T$ , then composing with the covering homomorphism  $H_1 \rightarrow H$ , we get a character on  $T_1$  and hence  $\mathcal{F} \subset \mathcal{F}_1$  and  $\mathcal{F}_H \subset \mathcal{F}_{H_1}$ . If  $\mu \in \mathcal{F}_H$ , then  $\mu - \delta_{\mathfrak{p}} \in \mathcal{F}$  and therefore we have that

$$\mathcal{F}_H \subset \mathcal{F}'_{H_1}. \quad (6.3)$$

Suppose that  $\mu \in \mathcal{F}'_{H_1}$  and let  $\lambda = \mu - \delta_{\mathfrak{p}} \in \mathcal{F}$ . Let  $(V_\mu, \tau_\mu)$  be the irreducible representation of  $H_1$  with highest weight  $\mu$ . This gives the representation  $(S \otimes V_\mu, \chi \otimes \tau_\mu)$  and we use this to form the induced bundle  $\underline{S \otimes V_\mu}$  on  $G_1/H_1$ . As in remark 3.2 of [Par72] we now show that  $\chi \otimes \tau_\mu$  actually is a representation of  $H$ . Any weight  $\eta$  of  $\chi \otimes \tau_\mu$  is of the form  $\eta = \gamma + \nu$  where  $\gamma$  and  $\nu$  are weights of  $\chi$  and  $\tau_\mu$  respectively. Hence

$$\eta = \delta_{\mathfrak{p}} - \sum_{\alpha \in \Phi} \alpha + \mu - \sum_{\alpha \in \Delta_{\mathfrak{h}}^+} m_\alpha \alpha$$

where  $m_\alpha \in \{0, 1, \dots\}$  and  $\Phi \subset \Delta_{\mathfrak{p}}^+$ . Therefore

$$\eta = \lambda + 2\delta_{\mathfrak{p}} - \sum_{\alpha \in \Delta^+} n_\alpha \alpha$$

where  $n_\alpha \in \{0, 1, \dots\}$ . Since each of the terms above gives rise to a character on  $T$ ,  $\chi \otimes \tau_\mu$  is a representation of  $H$ . So the following diagram commutes

$$\begin{array}{ccc} G_1 \times_{H_1} (S \otimes V_\mu) & \xrightarrow{\cong} & G \times_H (S \otimes V_\mu) \\ \downarrow \pi' & & \downarrow \pi \\ G_1/H_1 & \xrightarrow{\cong} & G/H \end{array}$$

where  $\pi$  and  $\pi'$  denote the usual projections. Hence we may identify the sections of  $G_1/H_1$ , (i.e., the elements of  $C^\infty(G_1) \otimes S \otimes V_\mu$  that are invariant under the representation  $\tilde{r} \otimes \chi \otimes \tau_\mu$  of  $H_1$ ) with the sections of  $G/H$ , (i.e., the elements of  $C^\infty(G) \otimes S \otimes V_\mu$  that are invariant under the representation  $\tilde{r} \otimes \chi \otimes \tau_\mu$  of  $H$ ). Under this identification, the left regular action  $\tilde{l}_\mu$  of  $G_1$  on  $\Gamma(\underline{S \otimes V_\mu})$  corresponds to the left regular action of  $G$  on  $\Gamma(\underline{S \otimes V_\mu})$  which commutes with the Dirac operator  $D_\mu: \Gamma(\underline{S \otimes V_\mu}) \rightarrow \Gamma(\underline{S \otimes V_\mu})$ . We also denote this action by  $\tilde{l}_\mu$ . Note that the differential of both actions  $\tilde{l}_\mu$  gives the left regular action of  $\mathfrak{g}$  on  $\Gamma(\underline{S \otimes V_\mu})$ . In a similar way we use the representations  $\chi^\pm \otimes \tau_\mu$  to get the Dirac operators  $D_\mu^\pm: \Gamma(\underline{S^\pm \otimes V_\mu}) \rightarrow \Gamma(\underline{S^\mp \otimes V_\mu})$  and the left regular action of  $G_1$  on  $\Gamma(\underline{S^\pm \otimes V_\mu})$  give actions  $\tilde{l}_\mu^\pm$  of  $G$  that commute with  $D_\mu^\pm$ .

We can now reformulate proposition 6.3 as follows.

**Proposition 6.4.** *Let  $G$  and  $H$  be compact connected Lie groups of equal rank where  $G$  is semisimple and  $G/H$  is a symmetric space. For  $\mu \in \mathcal{F}'_{H_1}$  let  $(V_\mu, \tau_\mu)$  be the irreducible representation of  $H_1$  with highest weight  $\mu$ . Let  $\lambda = \mu - \delta_{\mathfrak{p}}$ . Then the operator  $D_\mu^2: \Gamma(\underline{S \otimes V_\mu}) \rightarrow \Gamma(\underline{S \otimes V_\mu})$  is given by*

$$D_\mu^2 = (\tilde{l}_\mu)_*(\Omega) - \langle \lambda + 2\delta, \lambda \rangle 1$$

where  $(\tilde{l}_\mu)_*(\Omega)$  denotes the left regular action of the Casimir element of  $\mathfrak{g}$  on  $\Gamma(\underline{S \otimes V_\mu})$ .

Note that in the case  $G_1 = G$ ,  $H_1 = H$ , we have that  $\mathcal{F}'_{H_1} = \mathcal{F}_H$  and so in this case, proposition 6.4 is exactly proposition 6.3.

The following result, which is a direct consequence of proposition 6.4, is an important step in determining the representations  $\tilde{\pi}_\mu^\pm$  on the kernel of  $D_\mu^\pm$  in section 6.3.

**Corollary 6.5.** *Let  $\mu \in \mathcal{F}'_{H_1}$  and  $\lambda = \mu - \delta_{\mathfrak{p}}$ . If  $\pi_\nu$  is an irreducible subrepresentation of  $\tilde{\pi}_\mu^+$  or  $\tilde{\pi}_\mu^-$  with highest weight  $\nu$ , then*

$$\langle \nu + 2\delta, \nu \rangle = \langle \lambda + 2\delta, \lambda \rangle.$$

*Proof.* Suppose that  $\pi_\nu$  is an irreducible subrepresentation of  $\tilde{\pi}_\mu^+$  or  $\tilde{\pi}_\mu^-$  with highest weight  $\nu$ . Then since  $\ker D_\mu^\pm \subset \ker D_\mu^2$ , we must have that

$$\pi_\nu(\Omega) - \langle \lambda + 2\delta, \lambda \rangle = 0.$$

On the other hand, proposition 5.28 of [Kna02] shows that

$$\pi_\nu(\Omega) = \langle \nu + 2\delta, \nu \rangle.$$

We conclude that  $\nu$  must satisfy the equation

$$\langle \nu + 2\delta, \nu \rangle = \langle \lambda + 2\delta, \lambda \rangle.$$

□

### 6.3 The representations $\tilde{\pi}_\mu^\pm$

Recall that when we have assumptions as in proposition 6.4, the left regular action  $\tilde{l}_\mu^\pm$  of  $G$  on  $\Gamma(\underline{S^\pm \otimes V_\mu})$  restricts to representations  $\tilde{\pi}_\mu^\pm$  on  $\ker D_\mu^\pm$ . In the rest of this section  $G, H, G_1, H_1, \mu$  and  $\lambda$  will be as in proposition 6.4. The main result of this paper is as follows.

**Theorem 6.6.** *Let  $\mu \in \mathcal{F}'_{H_1}$ , i.e.,  $\mu \in \mathcal{F}_{H_1}$  and  $\lambda = \mu - \delta_{\mathfrak{p}} \in \mathcal{F}$ , and let  $\dim \mathfrak{p} = 2m$ .*

For  $m$  even we have that

$$\begin{aligned} \tilde{\pi}_\mu^+ &= \pi_{\sigma^{-1}(\lambda+\delta)-\delta} \text{ and } \tilde{\pi}_\mu^- = 0 & \text{if } \sigma^{-1}(\lambda+\delta) - \delta \in \mathcal{F}_G, \sigma \in W_1^+ \\ \tilde{\pi}_\mu^- &= \pi_{\sigma^{-1}(\lambda+\delta)-\delta} \text{ and } \tilde{\pi}_\mu^+ = 0 & \text{if } \sigma^{-1}(\lambda+\delta) - \delta \in \mathcal{F}_G, \sigma \in W_1^- \\ \tilde{\pi}_\mu^+ &= \tilde{\pi}_\mu^- = 0 & \text{otherwise.} \end{aligned}$$

For  $m$  odd we have that

$$\begin{aligned} \tilde{\pi}_\mu^- &= \pi_{\sigma^{-1}(\lambda+\delta)-\delta} \text{ and } \tilde{\pi}_\mu^+ = 0 & \text{if } \sigma^{-1}(\lambda+\delta) - \delta \in \mathcal{F}_G, \sigma \in W_1^+ \\ \tilde{\pi}_\mu^+ &= \pi_{\sigma^{-1}(\lambda+\delta)-\delta} \text{ and } \tilde{\pi}_\mu^- = 0 & \text{if } \sigma^{-1}(\lambda+\delta) - \delta \in \mathcal{F}_G, \sigma \in W_1^- \\ \tilde{\pi}_\mu^+ &= \tilde{\pi}_\mu^- = 0 & \text{otherwise.} \end{aligned}$$

Here  $\pi_{\sigma^{-1}(\lambda+\delta)+\delta}$  denotes the irreducible representation of  $G$  with highest weight  $\sigma^{-1}(\lambda+\delta)+\delta$  and  $W_1^\pm$  are as in proposition 5.8.

Now since for any  $\nu \in \mathcal{F}_G$  we have that  $\nu \in \mathcal{F}_H \subset \mathcal{F}'_{H_1}$  (see (6.3)) and

$$(\nu + \delta) - \delta = \nu \in \mathcal{F}_G,$$

theorem 6.6 characterizes all irreducible representations of  $G$  as representations in the kernel of certain  $D^+$  or  $D^-$ . We prove theorem 6.6 as follows.

**Step 1:** We study how  $\tilde{l}_\mu^\pm$  breaks up into irreducible parts.

**Step 2:** We prove proposition 6.10 which enables us to determine the multiplicities of some of the irreducible parts of  $\tilde{l}_\mu^\pm$ .

**Step 3:** Corollary 6.5, which is a direct consequence of the expression for  $D_\mu^2$ , gives a criterion which must be satisfied by the subrepresentations of  $\tilde{\pi}_\mu^\pm$ . This shows that the irreducible parts of  $\tilde{\pi}_\mu^\pm$  can have multiplicity at most one and that  $\tilde{\pi}_\mu^+$  and  $\tilde{\pi}_\mu^-$  can have no irreducible parts in common.

**Step 4:** We complete the proof of theorem 6.6 by combining step 3 with proposition 6.11 which shows that except for possibly one irreducible part, the representations  $\tilde{\pi}_\mu^+$  and  $\tilde{\pi}_\mu^-$  have all irreducible parts in common.

STEP 1. Consider the actions  $\tilde{l}_\mu^\pm$ . Extending  $\tilde{l}_\mu^\pm$  to the square integrable sections  $L_2(\underline{S \otimes V})$  of  $\underline{S \otimes V}$ , the Peter-Weyl theorem (section 3 of Bott [Bot65]) shows that

$$\tilde{l}_\mu^\pm = \sum_{\nu \in \mathcal{F}_G} \dim \text{Hom}_G(V_\nu, \Gamma(\underline{S^\pm \otimes V_\mu})) \pi_\nu$$

where  $(\pi_\nu, V_\nu)$  denotes the irreducible representation of  $G$  with highest weight  $\nu$ ,  $\text{Hom}_G(V, V')$  denotes the continuous linear maps respecting the action of  $G$  on  $V, V'$  and the sum is a unitary direct sum of representations. Frobenius reciprocity (proposition 2.1 of [Bot65]) shows that

$$\begin{aligned} \text{Hom}_G(V_\nu, \Gamma(\underline{S^\pm \otimes V_\mu})) &= \text{Hom}_H(V_\nu, S^\pm \otimes V_\mu) \\ &= \text{Hom}_{H_1}(V_\nu \otimes (S^\pm)^*, V_\mu) \end{aligned} \quad (6.4)$$



where  $(S^\pm)^*$  denotes the duals of  $S^\pm$ . Note that  $\text{Hom}_G(V_\nu, \Gamma(\underline{S^\pm \otimes V_\mu})) < \infty$ . We combine this with the following lemma.

**Lemma 6.7.** *Let  $\dim \mathfrak{p} = 2m$  and let  $(S^\pm)^*$  be the duals of  $S^\pm$ . As representations of  $H_1$  under the contragradient representations  $(\chi^\pm)^c$  we have that*

$$(S^\pm)^* = \begin{cases} S^\pm & \text{if } m \text{ is even} \\ S^\mp & \text{if } m \text{ is odd} \end{cases}.$$

*Proof.* Let  $\langle \cdot, \cdot \rangle_S$  be the inner product on  $S$  with respect to which  $\chi$  is unitary. We identify  $S^*$  with  $S$  through the linear anti-isomorphism  $*$ :  $S \rightarrow S^*$  given by

$$s^*(s') = \langle s', s \rangle_S \quad \text{for all } s, s' \in S.$$

Hence  $s \mapsto s^*$  is an isomorphism  $\overline{S} \rightarrow S^*$  where  $\overline{S}$  denotes the vector space  $S$  but where scalar multiplication is given by multiplication with the conjugate scalars. The contragradient representation  $\chi^c$  is given by

$$\begin{aligned} \chi^c(h)(s^*)(s') &= s^*(\chi(h^{-1})s') = \langle \chi(h^{-1})s', s \rangle_S \\ &= \langle s', \chi(h)s \rangle_S = (\chi(h)s)^*(s') \end{aligned}$$

for  $s, s' \in S, h \in H_1$ .

Recall that  $S$  is the unique representation on  $\text{Spin}(n)$  which is the restriction of an irreducible representation of the Clifford algebra  $\mathbb{C}l(n)$ . Since  $\overline{S}$  as a representation of  $\mathbb{C}l(n)$  is irreducible, we must have that  $\overline{S} = S$  as representations of  $\text{Spin}(n)$ . So it suffices to show that

$$\overline{S^\pm} = \begin{cases} S^\pm & \text{if } m \text{ is even} \\ S^\mp & \text{if } m \text{ is odd} \end{cases}. \quad (6.5)$$

Recall that  $S^\pm$  are the  $\pm 1$  eigenspaces of  $\sigma(\omega'_\mathbb{C})$  where

$$\omega'_\mathbb{C} = i^m X_1 \cdots X_{2m}.$$

Now since for  $s \in S$

$$\overline{\sigma(\omega'_\mathbb{C})s} = \begin{cases} \sigma(\omega'_\mathbb{C})s & \text{if } m \text{ is even} \\ -\sigma(\omega'_\mathbb{C})s & \text{if } m \text{ is odd} \end{cases},$$

we see that (6.5) holds. □

In the following  $\dim \mathfrak{p} = 2m$ . We conclude that

$$\tilde{l}_\mu^\pm = \begin{cases} \sum_{\nu \in \mathcal{F}_G} \dim \text{Hom}_{H_1}(S^\pm \otimes V_\nu, V_\mu) \pi_\nu & \text{if } m \text{ is even} \\ \sum_{\nu \in \mathcal{F}_G} \dim \text{Hom}_{H_1}(S^\mp \otimes V_\nu, V_\mu) \pi_\nu & \text{if } m \text{ is odd} \end{cases}. \quad (6.6)$$

STEP 2. Proposition 6.10 below (as in [Par72] lemma 8.1) enables us to determine the multiplicities of some of the irreducible parts  $\pi_\nu$  of  $\tilde{l}_\mu^\pm$ . In the proof of proposition 6.10 we use the following lemmas which can be found in Kostant [Kos61].

**Lemma 6.8.** *Let  $\pi_\delta$  be the irreducible representation of  $G_1$  with highest weight  $\delta$ . The weights of  $\pi_\delta$  are given by*

$$\rho_\Phi = \delta - \sum_{\alpha \in \Phi} \alpha$$

where  $\Phi \subset \Delta^+$ . The multiplicity of each  $\rho_\Phi$  is the number of ways in which  $\rho_\Phi$  can be expressed in this form.

*Proof.* See lemma 5.9 of [Kos61].

**Lemma 6.9.** *Let  $\nu_1, \nu_2 \in \mathcal{F}_{G_1}$  and let  $\pi_{\nu_1}, \pi_{\nu_2}$  denote the irreducible representations of  $G_1$  with highest weights  $\nu_1$  and  $\nu_2$  respectively. Suppose  $\xi_1$  is a weight of  $\pi_{\nu_1}$  and  $\xi_2$  is a weight of  $\pi_{\nu_2}$ . Then*

$$|\nu_1 + \nu_2| \geq |\xi_1 + \xi_2| \quad (6.7)$$

and equality holds exactly when

$$\nu_1 = w\xi_1, \quad \nu_2 = w\xi_2$$

for some  $w \in W$ .

*Proof.* (See lemma 5.8 of [Kos61]). If  $\nu_1 = w\xi_1, \nu_2 = w\xi_2$  for some  $w \in W$ , then clearly equality holds in (6.7). We now show the inequality (6.7) and that equality only holds if  $\nu_1 = w\xi_1, \nu_2 = w\xi_2$  for some  $w \in W$ . Let  $w \in W$  be such that

$$\langle w(\xi_1 + \xi_2), \alpha \rangle \geq 0 \quad \text{for all } \alpha \in \Delta^+. \quad (6.8)$$

Such a  $w$  exists according to corollary 2.68 of [Kna02]. Now let

$$\varphi_1 = \nu_1 - w\xi_1, \quad \varphi_2 = \nu_2 - w\xi_2.$$

Since  $w\xi_1$  is a weight of  $\pi_{\nu_1}$  and  $w\xi_2$  is a weight of  $\pi_{\nu_2}$  (see [Kna02] theorem 5.5(e)), we have that

$$w\xi_1 = \nu_1 - \sum_{\alpha \in \Delta^+} n_\alpha \alpha, \quad w\xi_2 = \nu_2 - \sum_{\alpha \in \Delta^+} m_\alpha \alpha$$

where  $n_\alpha, m_\alpha \in \{0, 1, \dots\}$ . Hence

$$\varphi_1 \geq 0, \quad \varphi_2 \geq 0. \quad (6.9)$$

Let

$$\varphi = \varphi_1 + \varphi_2 = \nu_1 + \nu_2 - w(\xi_1 + \xi_2).$$

Then  $\varphi \geq 0$  and we see that

$$|\nu_1 + \nu_2|^2 = |\xi_1 + \xi_2|^2 + |\varphi|^2 + 2\langle w(\xi_1 + \xi_2), \varphi \rangle.$$

Because of (6.8),  $\langle w(\xi_1 + \xi_2), \varphi \rangle \geq 0$ , and therefore we have proved the inequality (6.7). Furthermore, we see that if equality holds in (6.7), then  $|\varphi| = 0$ , i.e.,  $\varphi = 0$ . But by (6.9), this implies that  $\varphi_1 = \varphi_2 = 0$ , i.e.,  $\nu_1 = w\xi_1$  and  $\nu_2 = w\xi_2$ . This completes the proof.  $\square$

**Proposition 6.10.** *Let  $(\pi_\nu, V_\nu)$  be an irreducible representation of  $G$  with highest weight  $\nu \in \mathcal{F}_G$ . If  $\xi \in \mathcal{F}_{H_1}$  is the highest weight of an irreducible subrepresentation  $\tau_\xi$  of  $\chi \otimes \pi_\nu$ , then*

$$|\nu + \delta| \geq |\xi + \delta_{\mathfrak{h}}|$$

*and equality holds exactly when  $\xi$  is of the form*

$$\xi_\sigma = \sigma(\nu + \delta) - \delta_{\mathfrak{h}} \quad \text{for some } \sigma \in W_1.$$

*The multiplicity of  $\tau_{\xi_\sigma}$  in  $\chi \otimes \pi_\nu$  is one and  $\tau_{\xi_\sigma}$  is a subrepresentation of  $\chi^+ \otimes \pi_\nu$  when  $\sigma \in W_1^+$  and a subrepresentation of  $\chi^- \otimes \pi_\nu$  when  $\sigma \in W_1^-$  where  $W_1^\pm$  are as in proposition 5.8.*

*Proof.* According to [Kna02] proposition 9.72,  $\xi = \nu + \eta$  where  $\eta$  is a weight of  $\chi$ . Hence

$$\xi + \delta_{\mathfrak{h}} = \nu + \delta - \sum_{\alpha \in \Phi} \alpha$$

where  $\Phi \subset \Delta_{\mathfrak{p}}^+$ . Lemma 6.8 shows that  $\delta - \sum_{\alpha \in \Phi} \alpha$  is a weight of  $\pi_\delta$  where  $\pi_\delta$  denotes the irreducible representation of  $G_1$  with highest weight  $\delta$ . Since  $\nu$  is a weight of  $\pi_\nu$ , lemma 6.9 shows that  $|\nu + \delta| \geq |\xi + \delta_{\mathfrak{h}}|$  and equality holds if and only if  $\xi + \delta_{\mathfrak{h}} = w(\nu + \delta)$  for some  $w \in W$ . Suppose that such a  $w$  exists. Lemma 5.5 shows that there are unique  $s \in W_H, \sigma \in W_1$  such that  $w = s\sigma$ . We have that

$$\langle s\sigma(\nu + \delta) - \delta_{\mathfrak{h}}, \alpha \rangle = \langle \xi, \alpha \rangle \geq 0 \quad \text{for all } \alpha \in \Delta_{\mathfrak{h}}^+.$$

Proposition 2.69 of [Kna02] then shows that

$$\langle s\sigma(\nu + \delta), \alpha \rangle > 0 \quad \text{for all } \alpha \in \Delta_{\mathfrak{h}}^+. \quad (6.10)$$

Similarly,

$$\langle \nu + \delta, \alpha \rangle > 0 \quad \text{for all } \alpha \in \Delta^+$$

and therefore

$$\langle \sigma(\nu + \delta), \sigma\alpha \rangle > 0 \quad \text{for all } \alpha \in \Delta^+.$$

Since  $\Delta_{\mathfrak{h}}^+ \subset \sigma\Delta^+$  we have that

$$\langle \sigma(\nu + \delta), \alpha \rangle > 0 \quad \text{for all } \alpha \in \Delta_{\mathfrak{h}}^+. \quad (6.11)$$

Theorem 3.10.9 of [Wal73], (6.10) and (6.11) now shows that  $s = 1$ . We conclude that  $\xi = \xi_\sigma = \sigma(\nu + \delta) - \delta_{\mathfrak{h}}$ .

Finally, we show that the multiplicity of  $\tau_{\xi_\sigma}$  in  $\chi \otimes \pi_\nu$  is one and that  $\tau_{\xi_\sigma}$  is a subrepresentation of  $\chi^\pm \otimes \pi_\nu$  when  $\sigma \in W_1^\pm$ . Using proposition 9.72 of [Kna02] again, we get that the weights of any irreducible subrepresentation of  $\chi^\pm \otimes \pi_\nu$  are of the form  $\lambda^\pm + \gamma$  where  $\lambda^\pm$  is a weight of  $\chi^\pm$  and  $\gamma$  is a weight of  $\pi_\nu$ . In proposition 5.3 we saw that the weights of  $\chi$  are of the form  $\tilde{\lambda}_\varepsilon = \delta_{\mathfrak{p}} - \sum_{\alpha \in \Phi_\varepsilon} \alpha$  where  $\Phi_\varepsilon \subset \Delta_{\mathfrak{p}}^+$  and each of the weight spaces  $S_\varepsilon$  are one-dimensional. Hence

we can find a basis of  $S \otimes V_\nu$  consisting of elements  $s_\varepsilon \otimes v_\gamma$  where  $s_\varepsilon$  is a weight vector of  $\chi$  with weight  $\tilde{\lambda}_\varepsilon$  and  $v_\gamma$  is a weight vector of  $\pi_\nu$  with weight  $\gamma$ . So if we can show that  $\xi_\sigma = \sigma\delta - \delta_{\mathfrak{h}} + \sigma\nu = \tilde{\lambda}_\varepsilon + \gamma$  implies that  $\tilde{\lambda}_\varepsilon = \sigma\delta - \delta_{\mathfrak{h}}$  and  $\gamma = \sigma\nu$ , then we have proved the desired result. Suppose that  $\xi_\sigma = \tilde{\lambda}_\varepsilon + \gamma$ . Let

$$\rho = \tilde{\lambda}_\varepsilon + \delta_{\mathfrak{h}} = \delta - \sum_{\alpha \in \Phi_\varepsilon} \alpha.$$

This is a weight of  $\pi_\delta$  according to lemma 6.8 and we have that

$$\sigma(\nu + \delta) = \rho + \gamma.$$

Let

$$\psi_1 = \delta - \sigma^{-1}\rho, \quad \psi_2 = \nu - \sigma^{-1}\gamma.$$

Since by theorem 5.5 (e) of [Kna02],  $\sigma^{-1}\rho$  is a weight of  $\pi_\delta$  and  $\sigma^{-1}\gamma$  is a weight of  $\pi_\nu$ , we must have that

$$\sigma^{-1}\rho = \delta - \sum_{\alpha \in \Delta^+} n_\alpha \alpha, \quad \sigma^{-1}\gamma = \nu - \sum_{\alpha \in \Delta^+} m_\alpha \alpha$$

where  $n_\alpha, m_\alpha \in \{0, 1, \dots\}$  and therefore  $\psi_1 \geq 0, \psi_2 \geq 0$ . Since also

$$\psi_1 + \psi_2 = \nu + \delta - \sigma^{-1}(\rho + \gamma) = 0,$$

we conclude that  $\psi_1 = \psi_2 = 0$ . Hence  $\rho = \sigma\delta$  and  $\gamma = \sigma\nu$ , i.e.,  $\tilde{\lambda}_\varepsilon = \sigma\delta - \delta_{\mathfrak{h}}$  and  $\gamma = \sigma\nu$ .  $\square$

STEP 3. Now we turn to the subrepresentations  $\tilde{\pi}_\mu^\pm$  of  $\tilde{l}_\mu^\pm$ . Note that due to (6.6) we have that

$$\tilde{\pi}_\mu^\pm = \sum_{\nu \in \mathcal{F}_G} [\tilde{\pi}_\mu^\pm : \pi_\nu] \pi_\nu$$

where the multiplicity  $[\tilde{\pi}_\mu^\pm : \pi_\nu]$  of  $\pi_\nu$  in  $\tilde{\pi}_\mu^\pm$  for each  $\nu \in \mathcal{F}_G$  satisfies

$$[\tilde{\pi}_\mu^\pm : \pi_\nu] \leq \begin{cases} \dim \operatorname{Hom}_{H_1}(S^\pm \otimes V_\nu, V_\mu) & \text{if } m \text{ is even} \\ \dim \operatorname{Hom}_{H_1}(S^\mp \otimes V_\nu, V_\mu) & \text{if } m \text{ is odd} \end{cases}. \quad (6.12)$$

We are interested in the multiplicity of the representation  $\tau_\xi$  in  $\chi \otimes \pi_\nu$  when  $\xi = \mu = \lambda + \delta_{\mathfrak{p}}$  and  $\pi_\nu$  is a subrepresentation of  $\tilde{\pi}_\mu^+$  or  $\tilde{\pi}_\mu^-$ . Corollary 6.5 shows that in this case  $|\nu + \delta| = |\xi + \delta_{\mathfrak{h}}|$  since

$$\begin{aligned} |\nu + \delta|^2 - |(\lambda + \delta_{\mathfrak{p}}) + \delta_{\mathfrak{h}}|^2 &= |\nu + \delta|^2 - |\lambda + \delta|^2 \\ &= \langle \nu, \nu \rangle + \langle \delta, \delta \rangle + 2\langle \delta, \nu \rangle \\ &\quad - (\langle \lambda, \lambda \rangle + \langle \delta, \delta \rangle + 2\langle \delta, \lambda \rangle) \\ &= \langle \nu + 2\delta, \nu \rangle - \langle \lambda + 2\delta, \lambda \rangle = 0. \end{aligned}$$

Hence if  $\pi_\nu$  is a subrepresentation of  $\tilde{\pi}_\mu^+$ , proposition 6.10 and (6.12) show that

$$[\tilde{\pi}_\mu^+ : \pi_\nu] = 1 \text{ and } [\tilde{\pi}_\mu^- : \pi_\nu] = 0 \quad (6.13)$$

and if  $\pi_\nu$  is a subrepresentation of  $\tilde{\pi}_\mu^-$

$$[\tilde{\pi}_\mu^- : \pi_\nu] = 1 \text{ and } [\tilde{\pi}_\mu^+ : \pi_\nu] = 0. \quad (6.14)$$

STEP 4. We now show (proposition 6.11 below) that for each  $\nu \in \mathcal{F}_G$

$$[\tilde{\pi}_\mu^- : \pi_\nu] = [\tilde{\pi}_\mu^+ : \pi_\nu]$$

except possibly for one particular  $\nu_0 \in \mathcal{F}_G$ . Combining this with (6.13) and (6.14) shows that either

$$\begin{aligned} \tilde{\pi}_\mu^+ &= \tilde{\pi}_\mu^- = 0 \quad \text{or} \\ \tilde{\pi}_\mu^+ &= \pi_{\nu_0} \text{ and } \tilde{\pi}_\mu^- = 0 \quad \text{or} \\ \tilde{\pi}_\mu^- &= \pi_{\nu_0} \text{ and } \tilde{\pi}_\mu^+ = 0. \end{aligned}$$

This completes the proof of theorem 6.6.

**Proposition 6.11.** *Let  $\mu \in \mathcal{F}'_{H_1}$ ,  $\lambda = \mu - \delta_{\mathfrak{p}}$  and  $\dim \mathfrak{p} = 2m$ . Suppose that  $\sigma^{-1}(\lambda + \delta) - \delta \in \mathcal{F}_G$  for some  $\sigma \in W_1$ . Then  $\sigma$  is unique and*

$$\text{Tr } \tilde{\pi}_\mu^+ - \text{Tr } \tilde{\pi}_\mu^- = (-1)^m \text{sgn}(\sigma) \text{Tr } \pi_{\sigma^{-1}(\lambda + \delta) - \delta}.$$

If no such  $\sigma$  exists, then

$$\text{Tr } \tilde{\pi}_\mu^+ - \text{Tr } \tilde{\pi}_\mu^- = 0.$$

*Proof.* First we prove the uniqueness of  $\sigma$ . Let  $\sigma^{-1}(\lambda + \delta) - \delta \in \mathcal{F}_G$ . Then because of proposition 2.69 of [Kna02],

$$\langle \sigma^{-1}(\lambda + \delta), \alpha \rangle > 0 \quad \text{for } \alpha \in \Delta^+.$$

Theorem 3.10.9 of [Wal73] now shows that  $\sigma$  is unique.

For each  $\nu \in \mathcal{F}_G$  let

$$\Gamma_\nu^\pm = \{\varphi(v) \mid v \in V_\nu, \varphi \in \text{Hom}_G(V_\nu, \Gamma(\underline{S^\pm \otimes V_\mu}))\},$$

where  $(V_\nu, \pi_\nu)$  is the irreducible representation of  $G$  with highest weight  $\nu$ , i.e., we have that

$$L_2(\underline{S^\pm \otimes V_\mu}) = \sum_{\nu \in \mathcal{F}_G} \Gamma_\nu^\pm$$

where the sum is a unitary sum of representations under the left regular representation. Note that

$$\dim \Gamma_\nu^\pm = \dim V_\nu \cdot \dim \text{Hom}_G(V_\nu, \Gamma(\underline{S^\pm \otimes V_\mu})) < \infty.$$

Now let

$$D_\nu^\pm = D_\mu^\pm|_{\Gamma_\nu^\pm}.$$

We note that

$$\mathrm{Tr} \tilde{\pi}_\mu^+ - \mathrm{Tr} \tilde{\pi}_\mu^- = \sum_{\nu \in \mathcal{F}_G} \mathrm{ch}(\ker D_\nu^+) - \mathrm{ch}(\ker D_\nu^-)$$

where  $\mathrm{ch}(V)$  denotes the trace of the representation of  $G$  on  $V$  under the left regular action. Since for all  $v \in V_\mu$  and  $\varphi \in \mathrm{Hom}_G(V_\nu, \Gamma(\underline{S^\pm \otimes V_\mu}))$ ,

$$D_\nu^\pm \varphi(v) = \varphi'(v)$$

where  $\varphi' = D_\mu^\pm \circ \varphi \in \mathrm{Hom}_G(V_\nu, \Gamma(\underline{S^\mp \otimes V_\mu}))$ , we have that

$$D_\nu^\pm(\Gamma_\nu^\pm) \subset \Gamma_\nu^\mp$$

and  $D_\nu^\pm$  are the formal adjoints of each other. So we have that  $D_\nu^+ : \Gamma_\nu^+ \rightarrow \Gamma_\nu^-$  gives an isomorphism

$$(\ker D_\nu^+)^\perp \cong \mathrm{im} D_\nu^+ = (\ker D_\nu^-)^\perp$$

where  $\perp$  is taken within  $\Gamma_\nu^\pm$  with respect to the inner product on  $L_2(\underline{S^\pm \otimes V_\mu})$ . Therefore

$$\mathrm{ch}(\Gamma_\nu^\pm) = \mathrm{ch}((\ker D_\nu^\pm)^\perp) + \mathrm{ch}(\ker D_\nu^\pm) = \mathrm{ch}((\ker D_\nu^+)^\perp) + \mathrm{ch}(\ker D_\nu^\pm).$$

Hence using (6.4) we see that

$$\begin{aligned} \mathrm{ch}(\ker D_\nu^+) - \mathrm{ch}(\ker D_\nu^-) &= \mathrm{ch}(\Gamma_\nu^+) - \mathrm{ch}(\Gamma_\nu^-) \\ &= (-1)^m (\dim \mathrm{Hom}_{H_1}(S^+ \otimes V_\nu, V_\mu) \\ &\quad - \dim \mathrm{Hom}_{H_1}(S^- \otimes V_\nu, V_\mu)) \mathrm{Tr} \pi_\nu. \end{aligned}$$

In order to compute  $\dim \mathrm{Hom}_{H_1}(S^+ \otimes V_\nu, V_\mu) - \dim \mathrm{Hom}_{H_1}(S^- \otimes V_\nu, V_\mu)$  we calculate

$$\mathrm{Tr}(\chi^+ \otimes \pi_\nu) - \mathrm{Tr}(\chi^- \otimes \pi_\nu) = (\mathrm{Tr} \chi^+ - \mathrm{Tr} \chi^-) \mathrm{Tr} \pi_\nu \quad (6.15)$$

where we think of  $\pi_\nu$  as a representation on  $H_1$ . Using the Weyl character formula (theorem 5.75 of [Kna02]) and lemma 5.5 we get that

$$\begin{aligned} \mathrm{Tr} \pi_\nu &= \prod_{\alpha \in \Delta^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})^{-1} \sum_{w \in W} \mathrm{sgn}(w) e^{w(\nu + \delta)} \\ &= (\mathrm{Tr} \chi^+ - \mathrm{Tr} \chi^-)^{-1} \prod_{\alpha \in \Delta_\mathfrak{h}^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})^{-1} \sum_{\sigma \in W_1} \mathrm{sgn}(\sigma) \sum_{s \in W_H} \mathrm{sgn}(s) e^{s\sigma(\nu + \delta)}. \end{aligned}$$

Here we have used the expression for  $\mathrm{Tr} \chi^+ - \mathrm{Tr} \chi^-$  found in the proof of proposition 5.8. Now since  $\nu \in \mathcal{F}_G$ , we have that for each  $\sigma \in W_1$

$$\langle \nu + \delta, \alpha \rangle \geq \langle \delta, \alpha \rangle \quad \text{for all } \alpha \in \Delta^+$$

and therefore

$$\langle \sigma(\nu + \delta), \alpha \rangle \geq \langle \sigma\delta, \alpha \rangle \quad \text{for all } \alpha \in \sigma\Delta^+ \supset \Delta_{\mathfrak{h}}^+.$$

Hence

$$\langle \sigma(\nu + \delta) - \delta_{\mathfrak{h}}, \alpha \rangle \geq \langle \sigma\delta - \delta_{\mathfrak{h}}, \alpha \rangle = \langle \delta_{\mathfrak{p}}^\sigma, \alpha \rangle \geq 0 \quad \text{for all } \alpha \in \Delta_{\mathfrak{h}}^+,$$

i.e.,  $\sigma(\nu + \delta) - \delta_{\mathfrak{h}} \in \mathcal{F}_{H_1}$ . The Weyl character formula therefore shows that

$$\mathrm{Tr} \pi_\nu = (\mathrm{Tr} \chi^+ - \mathrm{Tr} \chi^-)^{-1} \sum_{\sigma \in W_1} \mathrm{sgn}(\sigma) \mathrm{Tr} \tau_{\sigma(\nu + \delta) - \delta_{\mathfrak{h}}}$$

where  $\tau_{\sigma(\nu + \delta) - \delta_{\mathfrak{h}}}$  denotes the irreducible representation of  $H_1$  with highest weight  $\sigma(\nu + \delta) - \delta_{\mathfrak{h}}$ . Inserting this into (6.15) we get

$$\mathrm{Tr}(\chi^+ \otimes \pi_\nu) - \mathrm{Tr}(\chi^- \otimes \pi_\nu) = \sum_{\sigma \in W_1} \mathrm{sgn}(\sigma) \mathrm{Tr} \tau_{\sigma(\nu + \delta) - \delta_{\mathfrak{h}}}.$$

Hence

$$\begin{aligned} & \dim \mathrm{Hom}_{H_1}(S^+ \otimes V_\nu, V_\mu) - \dim \mathrm{Hom}_{H_1}(S^- \otimes V_\nu, V_\mu) \\ &= \sum_{\sigma \in W_1} \mathrm{sgn}(\sigma) \dim \mathrm{Hom}_{H_1}(V_{\sigma(\nu + \delta) - \delta_{\mathfrak{h}}}, V_\mu). \end{aligned}$$

Therefore

$$\mathrm{Tr} \tilde{\pi}_\mu^+ - \mathrm{Tr} \tilde{\pi}_\mu^- = (-1)^m \sum_{\nu \in \mathcal{F}_G} \sum_{\sigma \in W_1} \mathrm{sgn}(\sigma) \dim \mathrm{Hom}_{H_1}(V_{\sigma(\nu + \delta) - \delta_{\mathfrak{h}}}, V_\mu).$$

Note that

$$\dim \mathrm{Hom}_{H_1}(V_{\sigma(\nu + \delta) - \delta_{\mathfrak{h}}}, V_\mu) = \begin{cases} 1 & \text{if } \sigma(\nu + \delta) - \delta_{\mathfrak{h}} = \mu \\ 0 & \text{otherwise} \end{cases}$$

and  $\sigma(\nu + \delta) - \delta_{\mathfrak{h}} = \mu = \lambda + \delta_{\mathfrak{p}}$  exactly when  $\sigma^{-1}(\lambda + \delta) - \delta = \nu$ . Since this can happen for at most one  $\sigma \in W_1$ , we conclude that

$$\mathrm{Tr} \tilde{\pi}_\mu^+ - \mathrm{Tr} \tilde{\pi}_\mu^- = \begin{cases} (-1)^m \mathrm{sgn}(\sigma) \mathrm{Tr} \pi_{\sigma^{-1}(\lambda + \delta) - \delta} & \text{if } \sigma^{-1}(\lambda + \delta) - \delta \in \mathcal{F}_G \\ 0 & \text{otherwise} \end{cases}.$$

□

*Remark 6.12.* Note that since  $\mathcal{F}_G$  is a lattice, only finitely many  $\nu \in \mathcal{F}_G$  can satisfy the equation

$$|\nu + \delta| = |\lambda + \delta|.$$

Hence  $\tilde{\pi}_\mu^\pm$  can only have finitely many subrepresentations  $\pi_\nu$ . This shows that in the symmetric case, the fact that the kernels  $\ker D_\mu^\pm$  of  $D_\mu^\pm$  are finite-dimensional follows directly from corollary 6.5, (i.e., without the ellipticity of  $D_\mu$ ).

## 7 Examples

In the following we study some specific examples. The notation is as in section 6.

### 7.1 $\mathrm{SO}(3)/\mathrm{SO}(2)$

Let  $G = \mathrm{SO}(3)$  and  $H = \mathrm{SO}(2)$  as in example 5.1. We now show that we can identify the representations  $\tilde{\pi}_\mu^+$  and  $\tilde{\pi}_\mu^-$  for all  $\mu \in \mathcal{F}'_{H_1}$  as in theorem 6.6. Although theorem 6.6 gives us the result we need, we will in this case do some more direct computations based on theorem 9.16 of [Kna02]. Using the same notation as in example 5.1 we get that

$$\delta = \delta_{\mathfrak{p}} = \frac{1}{2}e_1$$

and

$$W = W_1 = \{1, s_{e_1}\}$$

where  $s_{e_1}$  denotes reflection in the root  $e_1$ . Since  $\mathrm{SO}(\mathfrak{p}) = \mathrm{SO}(2) = H$ ,  $\mathrm{Ad}: H \rightarrow \mathrm{SO}(\mathfrak{p})$  is the standard representation and this lifts to the identity map  $H_1 = \mathrm{Spin}(2) \rightarrow \mathrm{Spin}(\mathfrak{p}) = \mathrm{Spin}(2)$ . We get the 1-dimensional spinor representations  $\chi^\pm$  of  $H_1$  with highest weights  $\pm \frac{1}{2}e_1$ . Since  $H = S^1$ , we must have that the analytically integral forms of  $H$  are exactly of the form  $\lambda_1 e_1$  where  $\lambda_1 \in \mathbb{Z}$ . Since  $H$  has no roots, we therefore have that

$$\mathcal{F} = \mathcal{F}_H = \{\lambda_1 e_1 \mid \lambda_1 \in \mathbb{Z}\}.$$

Note that

$$\mathcal{F}_G = \{\nu_1 e_1 \mid \nu_1 \in \mathbb{Z}, \nu_1 \geq 0\}.$$

Since  $H_1 = \mathrm{Spin}(2)$  is the two-fold cover of  $H = S^1$ , we have that

$$\mathcal{F}_1 = \mathcal{F}_{H_1} = \{\lambda_1 e_1 \mid \lambda_1 \in \mathbb{Z} \text{ or } \lambda_1 \in \mathbb{Z} + \frac{1}{2}\}.$$

Hence

$$\mathcal{F}'_{H_1} = \{\mu_1 e_1 \mid \mu_1 \in \mathbb{Z} + \frac{1}{2}\}.$$

**Proposition 7.1.** *For  $G = \mathrm{SO}(3)$  and  $H = \mathrm{SO}(2)$  let  $\mu \in \mathcal{F}'_{H_1}$  and  $\lambda = \mu - \delta_{\mathfrak{p}} = \lambda_1 e_1$ . We have that*

$$\begin{aligned} \tilde{\pi}_\mu^- &= \pi_\lambda \text{ and } \tilde{\pi}_\mu^+ = 0 & \text{for } \lambda_1 \geq 0 \\ \tilde{\pi}_\mu^+ &= \pi_{-(\lambda + e_1)} \text{ and } \tilde{\pi}_\mu^- = 0 & \text{for } \lambda_1 \leq -1 \end{aligned} .$$

*Proof.* Let  $(V_\mu, \tau_\mu)$  be an irreducible representation of  $H_1$  with highest weight  $\mu = \lambda + \delta_{\mathfrak{p}} = (\lambda_1 + \frac{1}{2})e_1$ . Then  $\dim V_\mu = 1$  and therefore  $\chi^\pm \otimes \tau_\mu$  are 1-dimensional and hence irreducible representations of  $H$  with highest weights  $(\pm \frac{1}{2} + \lambda_1 + \frac{1}{2})e_1$ . In the following we show that any irreducible component of  $\tilde{\pi}_\mu^\pm$  has multiplicity at most 1. Let  $\pi_\nu$  be an irreducible representation of  $G$  with



highest weight  $\nu = \nu_1 e_1 \in \mathcal{F}_G$ . Frobenius reciprocity (theorem 9.9 of [Kna02]) gives us that

$$[\tilde{\pi}_\mu^\pm : \pi_\nu] \leq [\tilde{l}_\mu^\pm : \pi_\nu] = [\pi_\nu|_H : \chi^\pm \otimes \tau_\mu].$$

Theorem 9.16 of [Kna02] shows that a restriction of an irreducible representation of  $G$  to  $H$  decomposes with multiplicity one into irreducible representations of  $H$ . Hence

$$[\tilde{\pi}_\mu^\pm : \pi_\nu] \leq 1.$$

Now we show that  $\tilde{\pi}_\mu^\pm$  has at most one irreducible component. If  $\pi_\nu$  is an irreducible subrepresentation of  $\tilde{\pi}_\mu^+$  or  $\tilde{\pi}_\mu^-$ , then corollary 6.5 shows that

$$(\nu_1^2 + \nu_1) - (\lambda_1^2 + \lambda_1) = (\nu_1 - \lambda_1)(\nu_1 + \lambda_1 + 1) = 0,$$

i.e.,

$$\nu_1 = \lambda_1 \quad \text{or} \quad \nu_1 = -(\lambda_1 + 1). \quad (7.1)$$

Using Frobenius reciprocity again we see that in order for  $[\tilde{\pi}_\mu^\pm : \pi_\nu]$  to be non-zero,  $\chi^\pm \otimes \tau_\mu$  must be an irreducible subrepresentation of  $\pi_\nu|_H$ . Recall that  $\chi^+ \otimes \tau_\mu$  has highest weight  $(\frac{1}{2} + \lambda_1 + \frac{1}{2})e_1 = (\lambda_1 + 1)e_1$  and  $\chi^- \otimes \tau_\mu$  has highest weight  $(-\frac{1}{2} + \lambda_1 + \frac{1}{2})e_1 = \lambda_1 e_1$ . Theorem 9.16 of [Kna02] shows that the highest weights of the irreducible subrepresentations of  $\pi_\nu|_H$  are exactly of the form  $\eta = \eta_1 e_1$  where  $|\eta_1| \leq \nu_1$ . So in order that  $[\tilde{\pi}_\mu^+ : \pi_\nu] \neq 0$  we must have that  $|\lambda_1 + 1| \leq \nu_1$  and in order that  $[\tilde{\pi}_\mu^- : \pi_\nu] \neq 0$  we must have that  $|\lambda_1| \leq \nu_1$ . Using (7.1) we therefore see that

$$\begin{aligned} [\tilde{\pi}_\mu^+ : \pi_\nu] &= 0 \quad \text{unless possibly when } \lambda_1 \leq -1 \text{ and } \nu_1 = -(\lambda_1 + 1) \\ [\tilde{\pi}_\mu^- : \pi_\nu] &= 0 \quad \text{unless possibly when } \lambda_1 \geq 0 \text{ and } \nu_1 = \lambda_1. \end{aligned}$$

We conclude that if  $\lambda_1 \geq 0$ , then

$$\begin{aligned} [\tilde{\pi}_\mu^+ : \pi_\nu] &= 0 \quad \text{for all } \nu \\ [\tilde{\pi}_\mu^- : \pi_\nu] &= 0 \quad \text{for } \nu \neq \lambda, \end{aligned}$$

and if  $\lambda_1 \leq -1$ , then

$$\begin{aligned} [\tilde{\pi}_\mu^+ : \pi_\nu] &= 0 \quad \text{for } \nu \neq -(\lambda + e_1) \\ [\tilde{\pi}_\mu^- : \pi_\nu] &= 0 \quad \text{for all } \nu. \end{aligned}$$

Observe that

$$\begin{aligned} 1(\lambda + \delta) - \delta &= \lambda \in \mathcal{F}_G && \text{for } \lambda_1 \geq 0 \\ s_{e_1}(\lambda + \delta) - \delta &= -(\lambda + e_1) \in \mathcal{F}_G && \text{for } \lambda_1 \leq -1 \end{aligned} \quad .$$

So proposition 6.11 shows that

$$\text{Tr } \tilde{\pi}_\mu^+ - \text{Tr } \tilde{\pi}_\mu^- = \begin{cases} -\pi_\lambda & \text{for } \lambda_1 \geq 0 \\ \pi_{-(\lambda + e_1)} & \text{for } \lambda_1 \leq -1 \end{cases} \quad .$$

We conclude that

$$\begin{aligned} \tilde{\pi}_\mu^- &= \pi_\lambda \text{ and } \tilde{\pi}_\mu^+ = 0 && \text{for } \lambda_1 \geq 0 \\ \tilde{\pi}_\mu^+ &= \pi_{-(\lambda + e_1)} \text{ and } \tilde{\pi}_\mu^- = 0 && \text{for } \lambda_1 \leq -1 \end{aligned} \quad .$$

□

## 7.2 $\mathrm{SO}(2m+1)/\mathrm{SO}(2m)$

As in example 5.2 let  $G = \mathrm{SO}(2m+1)$  and  $H = \mathrm{SO}(2m)$  where  $m \geq 2$ . We have that  $G_1 = \mathrm{Spin}(2m+1)$  and  $H_1 = \mathrm{Spin}(2m)$  are the universal covers of  $G$  and  $H$  respectively. As in the case of  $\mathrm{SO}(3)/\mathrm{SO}(2)$ , we use theorem 9.16 of [Kna02] to study the representations  $\tilde{\pi}_\mu^\pm$  for  $\mu \in \mathcal{F}'_{H_1}$ . Using the notation of example 5.2, we see that

$$\delta = \frac{1}{2} \sum_{k=1}^m (m-k+1)e_k, \quad \delta_{\mathfrak{p}} = \frac{1}{2} \sum_{k=1}^m e_k.$$

The simple roots of  $G$  are

$$e_1 - e_2, e_2 - e_3, \dots, e_{m-1} - e_m, e_m$$

and the simple roots of  $H$  are

$$e_1 - e_2, e_2 - e_3, \dots, e_{m-1} - e_m, e_{m-1} + e_m.$$

(See [Kna02] section 2.5). We now find the analytically integral forms  $\mathcal{F}$  and  $\mathcal{F}_1$ . Since  $G_1$  is compact, semisimple and has center  $\{\pm 1\}$ , proposition 4.68 of [Kna02] shows that the analytically integral forms  $\mathcal{F}_1$  are given by

$$\mathcal{F}_1 = \left\{ \sum_{k=1}^m \lambda_k e_k \mid \lambda_k \in \mathbb{Z} \text{ for all } k \text{ or } \lambda_k \in \mathbb{Z} + \frac{1}{2} \text{ for all } k \right\}.$$

Hence proposition 4.67 of [Kna02] shows that the analytically integral forms  $\mathcal{F}$  are given by

$$\mathcal{F} = \left\{ \sum_{k=1}^m \lambda_k e_k \mid \lambda_k \in \mathbb{Z} \text{ for all } k \right\}.$$

When checking dominance of an analytically integral form, it is enough to check the dominance with respect to simple roots. An element of  $\sum_{k=1}^m \nu_k e_k \in \mathcal{F}_1$  is dominant with respect to  $G$  exactly when

$$\nu_1 \geq \nu_2 \geq \dots \geq \nu_m \geq 0.$$

An element  $\mu = \sum_{k=1}^m \mu_k e_k \in \mathcal{F}_1$  is dominant with respect to  $H$  exactly when

$$\mu_1 \geq \mu_2 \geq \dots \geq |\mu_m|.$$

Hence

$$\mathcal{F}'_{H_1} = \left\{ \sum_{k=1}^m \mu_k e_k \mid \mu_k \in \mathbb{Z} + \frac{1}{2} \text{ for all } k \text{ and } \mu_1 - \frac{1}{2} \geq \mu_2 - \frac{1}{2} \geq \dots \geq |\mu_m - \frac{1}{2}| \right\}$$

We now prove the following.

**Proposition 7.2.** *Let  $G = \mathrm{SO}(2m+1)$  and  $H = \mathrm{SO}(2m)$  where  $m \geq 2$ . Let  $\mu \in \mathcal{F}'_{H_1}$  and let  $\lambda = \mu - \delta_{\mathfrak{p}} = \sum_{k=1}^m \lambda_k e_k$  and  $\lambda' = \sum_{k=1}^{m-1} \lambda_k e_k - (\lambda_m + 1)e_m$ . If  $m$  is even*

$$\begin{aligned} \tilde{\pi}_{\mu}^{+} &= \pi_{\lambda} \text{ and } \tilde{\pi}_{\mu}^{-} = 0 & \text{for } \lambda_m \geq 0 \\ \tilde{\pi}_{\mu}^{-} &= \pi_{\lambda'} \text{ and } \tilde{\pi}_{\mu}^{+} = 0 & \text{for } \lambda_m \leq -1 \end{aligned}$$

and if  $m$  is odd

$$\begin{aligned} \tilde{\pi}_{\mu}^{-} &= \pi_{\lambda} \text{ and } \tilde{\pi}_{\mu}^{+} = 0 & \text{for } \lambda_m \geq 0 \\ \tilde{\pi}_{\mu}^{+} &= \pi_{\lambda'} \text{ and } \tilde{\pi}_{\mu}^{-} = 0 & \text{for } \lambda_m \leq -1 \end{aligned} .$$

*Proof.* Let  $(V_{\mu}, \tau_{\mu})$  be the irreducible representation of  $H_1 = \mathrm{Spin}(n)$  with highest weight

$$\mu = \lambda + \delta_{\mathfrak{p}} = \sum_{k=1}^m (\lambda_k + \frac{1}{2}) e_k.$$

The half spinor representations  $\chi^{\pm}$  of  $H_1$  have weights

$$\frac{1}{2} \sum_{k=1}^m \varepsilon_k e_k \text{ where } \varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{\pm 1\}^m$$

and such a weight is a weight of  $\chi^{+}$  exactly when  $\varepsilon_k = -1$  for an even number of  $\varepsilon_k$ . Proposition 9.72 of [Kna02] shows that the highest weights of the irreducible parts of  $\chi^{\pm} \otimes \tau_{\mu}$  have the form

$$\eta = \sum_{k=1}^m (\lambda_k + \frac{1}{2}(1 + \varepsilon_k)) e_k \text{ where } \varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{\pm 1\}^m$$

with an even number of  $\varepsilon_k = -1$  if  $\eta$  is a weight of  $\chi^{+} \otimes \tau_{\mu}$  and an odd number of  $\varepsilon_k = -1$  if  $\eta$  is a weight of  $\chi^{-} \otimes \tau_{\mu}$ . Note that

$$\eta_k = \begin{cases} \lambda_k & \text{if } \varepsilon_k = -1 \\ \lambda_k + 1 & \text{if } \varepsilon_k = 1 \end{cases} .$$

Let  $\pi_{\nu}$  be an irreducible representation of  $G$  with highest weight  $\nu$ , i.e.  $\nu = \sum_{k=1}^m \nu_k e_k \in \mathcal{F}_G$  where  $\nu_k \in \mathbb{Z}$  for all  $k$  and  $\nu_1 \geq \nu_2 \geq \dots \geq \nu_m \geq 0$ . Suppose that  $\pi_{\nu}$  is a subrepresentation of either  $\tilde{\pi}_{\mu}^{+}$  or  $\tilde{\pi}_{\mu}^{-}$ . Then corollary 6.5 shows that

$$\sum_{k=1}^m (\nu_k^2 + (m - k + 1)\nu_k) = \sum_{k=1}^m (\lambda_k^2 + (m - k + 1)\lambda_k). \quad (7.2)$$

The irreducible part  $\gamma_{\eta^{\pm}}$  of  $\chi^{\pm} \otimes \tau_{\mu}$  with highest weight  $\eta^{\pm}$  induces a subrepresentation of  $\tilde{l}_{\mu}^{\pm}$ . Now let  $\tilde{\gamma}_{\eta^{\pm}}$  be the restriction of this subrepresentation to the kernel of  $D_{\mu}^{\pm}$ . Note that

$$[\tilde{\pi}_{\mu}^{+} : \pi_{\nu}] = \sum_{\eta^{+}} [\tilde{\gamma}_{\eta^{+}} : \pi_{\nu}], \quad [\tilde{\pi}_{\mu}^{-} : \pi_{\nu}] = \sum_{\eta^{-}} [\tilde{\gamma}_{\eta^{-}} : \pi_{\nu}].$$

Frobenius reciprocity now shows that

$$[\tilde{\gamma}_{\eta^\pm} : \pi_\nu] \leq [\pi_\nu|_H : \gamma_{\eta^\pm}].$$

Theorem 9.16 of [Kna02] shows that the irreducible parts of  $\pi_\nu|_H$  have multiplicity one and are exactly the representations of  $H$  with highest weights  $\sum_{k=1}^m a_k e_k$  satisfying

$$\nu_1 \geq a_1 \geq \nu_2 \geq a_2 \geq \cdots \geq \nu_m \geq |a_m|.$$

So in order for  $\pi_\nu$  to be a subrepresentation of  $\tilde{\pi}_\mu^+$ , we must have that

$$\nu_1 \geq \lambda_1 + \frac{1}{2}(1 + \varepsilon_1) \geq \cdots \geq \nu_m \geq |\lambda_m + \frac{1}{2}(1 + \varepsilon_m)| \quad (7.3)$$

for some  $\varepsilon$  with an even number of  $\varepsilon_k = -1$  and in order for  $\pi_\nu$  to be a subrepresentation of  $\tilde{\pi}_\mu^-$ , we must have that

$$\nu_1 \geq \lambda_1 + \frac{1}{2}(1 + \varepsilon_1) \geq \cdots \geq \nu_m \geq |\lambda_m + \frac{1}{2}(1 + \varepsilon_m)| \quad (7.4)$$

for some  $\varepsilon$  with an odd number of  $\varepsilon_k = -1$ .

Suppose that  $\lambda_m \geq 0$ , (i.e.,  $\lambda \in \mathcal{F}_G$ ). For any  $\nu$  satisfying (7.3) or (7.4), we have in particular that

$$\nu_k \geq \lambda_k \geq 0 \quad \text{for } k \in \{1, \dots, m\}.$$

When  $\nu$  also satisfies (7.2), then (7.3) or (7.4) can only hold if

$$\varepsilon_1 = \cdots = \varepsilon_m = -1 \text{ and } \nu_k = \lambda_k \text{ for } k \in \{1, \dots, m\}.$$

So we have that

$$\begin{aligned} [\tilde{\gamma}_{\eta^\pm} : \pi_\nu] &= 0 && \text{for } \nu \neq \lambda \\ [\tilde{\gamma}_{\eta^\pm} : \pi_\lambda] &= 0 && \text{unless possibly for one } \eta^+ \text{ if } m \text{ is even} \\ [\tilde{\gamma}_{\eta^\pm} : \pi_\lambda] &= 0 && \text{unless possibly for one } \eta^- \text{ if } m \text{ is odd.} \end{aligned}$$

Hence

$$\begin{aligned} [\tilde{\pi}_\mu^\pm : \pi_\nu] &= 0 && \text{for } \nu \neq \lambda \\ [\tilde{\pi}_\mu^+ : \pi_\lambda] &\leq 1, [\tilde{\pi}_\mu^- : \pi_\lambda] = 0 && \text{if } m \text{ is even} \\ [\tilde{\pi}_\mu^- : \pi_\lambda] &\leq 1, [\tilde{\pi}_\mu^+ : \pi_\lambda] = 0 && \text{if } m \text{ is odd.} \end{aligned}$$

Now using proposition 6.11, we see that since  $\lambda = 1(\lambda + \delta) - \delta \in \mathcal{F}_G$ , the representations  $\tilde{\pi}_\mu^\pm$  cannot both be zero. We conclude that

$$\begin{aligned} \tilde{\pi}_\mu^+ &= \pi_\lambda \text{ and } \tilde{\pi}_\mu^- = 0 && \text{if } m \text{ is even} \\ \tilde{\pi}_\mu^- &= \pi_\lambda \text{ and } \tilde{\pi}_\mu^+ = 0 && \text{if } m \text{ is odd} \end{aligned} \quad .$$

Suppose that  $\lambda_m \leq -1$ . If  $\nu$  satisfies (7.3) or (7.4), we have in particular that

$$\nu_k \geq \lambda_k \geq 0 \text{ for } k \in \{1, \dots, m-1\}, \quad \nu_m \geq |\lambda_m| - 1 = -(\lambda_m + 1). \quad (7.5)$$

The condition (7.2) is given by

$$\sum_{k=1}^{m-1} ((\nu_k^2 - \lambda_k^2) + (m - k + 1)(\nu_k - \lambda_k)) + ((\nu_m^2 - \lambda_m^2) + (\nu_m - \lambda_m)) = 0. \quad (7.6)$$

Now (7.5) implies that the first term of (7.6) is non-negative and therefore that the second term is non-positive. The second term is a second order polynomial in  $\nu_m$  with zeroes at  $\lambda_m$  and  $-(\lambda_m + 1)$  so it is non-positive if  $\nu_m$  lies in the interval  $[\lambda_m, -(\lambda_m + 1)]$ . Since also  $\nu_m \in \{-(\lambda_m + 1), -\lambda_m, \dots\}$  we conclude that for (7.3) or (7.4) and (7.2) to hold we must have that

$$\varepsilon_k = -1, \nu_k = \lambda_k \text{ for } k \in \{1, \dots, m-1\} \quad \text{and} \quad \varepsilon_m = 1, \nu_m = -(\lambda_m + 1).$$

Hence if  $\lambda' = \sum_{k=1}^{m-1} \lambda_k e_k - (\lambda_m + 1)e_m$ , we have that

$$\begin{aligned} [\tilde{\gamma}_{\eta^\pm} : \pi_\nu] &= 0 & \text{for } \nu \neq \lambda' \\ [\tilde{\gamma}_{\eta^\pm} : \pi_{\lambda'}] &= 0 & \text{unless possibly for one } \eta^- \text{ if } m \text{ is even} \\ [\tilde{\gamma}_{\eta^\pm} : \pi_{\lambda'}] &= 0 & \text{unless possibly for one } \eta^+ \text{ if } m \text{ is odd} \end{aligned}$$

and

$$\begin{aligned} [\tilde{\pi}_\mu^\pm : \pi_\nu] &= 0 & \text{for } \nu \neq \lambda' \\ [\tilde{\pi}_\mu^- : \pi_{\lambda'}] &\leq 1, [\tilde{\pi}_\mu^- : \pi_\lambda] = 0 & \text{if } m \text{ is even} \\ [\tilde{\pi}_\mu^+ : \pi_{\lambda'}] &\leq 1, [\tilde{\pi}_\mu^+ : \pi_\lambda] = 0 & \text{if } m \text{ is odd.} \end{aligned}$$

Now using proposition 6.11, we see that since  $\lambda' = s_{e_m}(\lambda + \delta) - \delta \in \mathcal{F}_G$ , the representations  $\tilde{\pi}_\mu^\pm$  cannot both be zero. We conclude that

$$\begin{aligned} \tilde{\pi}_\mu^- &= \pi_{\lambda'} \text{ and } \tilde{\pi}_\mu^+ = 0 & \text{if } m \text{ is even} \\ \tilde{\pi}_\mu^+ &= \pi_{\lambda'} \text{ and } \tilde{\pi}_\mu^- = 0 & \text{if } m \text{ is odd} \end{aligned} \quad .$$

□

## References

- [Bot65] Raoul Bott. The index theorem for homogeneous differential operators. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 167–186. Princeton Univ. Press, Princeton, N.J., 1965.
- [Dup03] Johan Dupont. *Fibre bundles and Chern–Weil Theory*. Unpublished course notes from Århus Universitet, 2003.
- [GM91] John E. Gilbert and Margaret A. M. Murray. *Clifford algebras and Dirac operators in harmonic analysis*, volume 26 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991.
- [Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996.
- [Kna02] Anthony W. Knaapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhauser Boston Inc., Boston, MA, 2002.
- [Kos61] Bertram Kostant. Lie algebra cohomology and the generalized Borel–Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.
- [LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [Par72] R. Parthasarathy. Dirac operator and the discrete series. *Ann. of Math. (2)*, 96:1–30, 1972.
- [Sle87] Stephen Slebarski. The Dirac operator on homogeneous spaces and representations of reductive Lie groups. I. *Amer. J. Math.*, 109(2):283–301, 1987.
- [Wal73] Nolan R. Wallach. *Harmonic analysis on homogeneous spaces*. Marcel Dekker Inc., New York, 1973.
- [War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983.